

Chapter 7

Oscillations

- **Simple harmonic oscillation**
- **Coupled oscillation**
- **Damped oscillation**
- * **Nonlinear oscillation**
- **Forced oscillation under friction**

7.1 Simple Harmonic motion

$$F = -kx$$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \omega_0^2 x = 0, \quad \omega_0^2 = \frac{k}{m}$$

$$x = A \cos \omega t + B \sin \omega t$$

$$x = A \cos(\omega t + \varphi)$$

$$x = Ae^{i\omega t} + Be^{-i\omega t}$$

Under initial conditions

$$x(0) = x_0 \quad \dot{x}(0) = \dot{x}_0$$

the solution is

$$\left\{ \begin{array}{l} x = A \cos(\omega t + \varphi) \\ A = \left[x_0^2 + \frac{\dot{x}_0^2}{\omega_0^2} \right]^{1/2}, \quad \varphi = \arctan \left[-\frac{\dot{x}_0}{\omega x_0} \right] \\ \omega = \omega_0 = \sqrt{k/m} \end{array} \right.$$

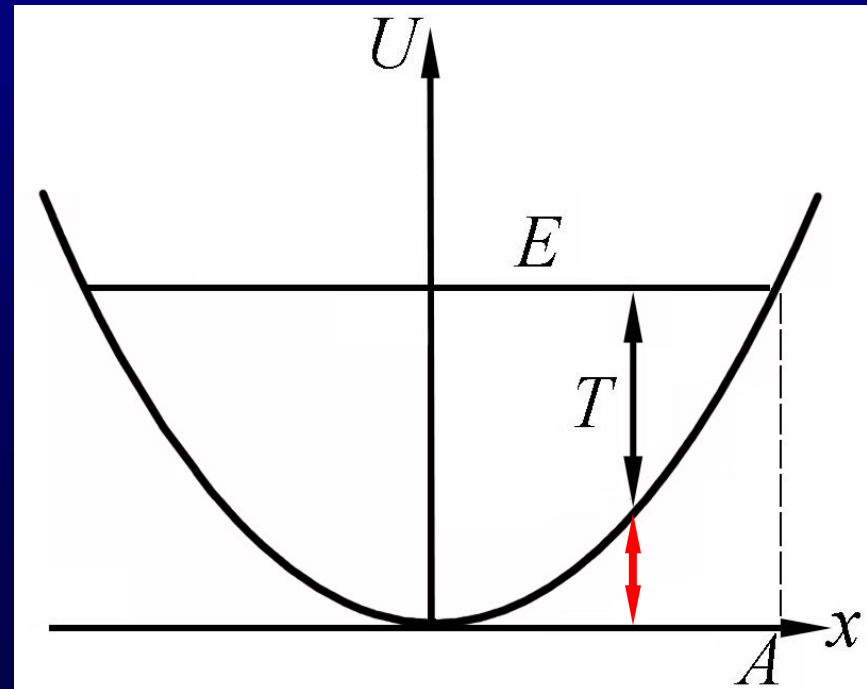
The first integral

$$m\ddot{x} = -kx \quad \dot{x}dt = dx$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m\dot{x}_0^2 + \frac{1}{2}m\omega_0^2 x_0^2 = E$$

Potential energy curve

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2$$



Time or spatial average

$$\langle P \rangle_T = \frac{1}{T} \int_0^T P dt$$

$$\langle P \rangle_A = \frac{1}{A} \int_0^A P dx$$

Same physics, same equation

* LC circuit

$$\frac{1}{2}LI^2 + \frac{1}{2C}q^2 = E$$

$$L\frac{dI}{dt} + \frac{1}{C}q = 0$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

$$\omega^2 = \frac{1}{LC}$$

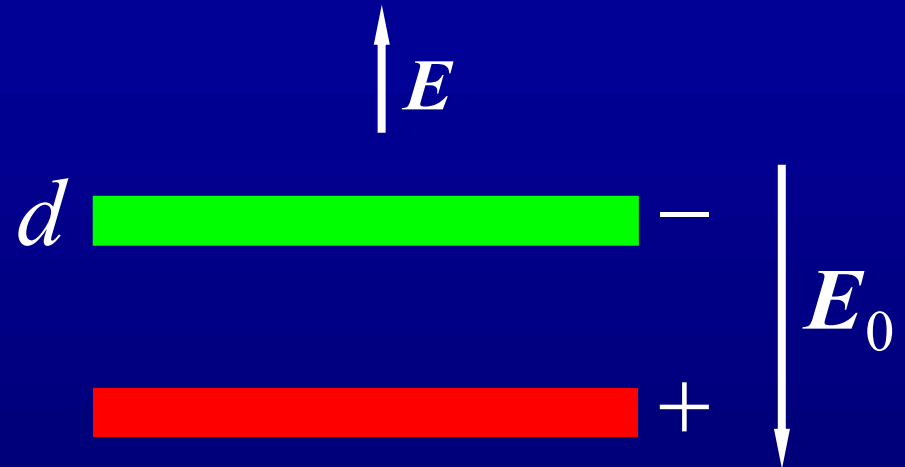
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$



Plasma oscillation

$$E = \frac{q}{\epsilon_0 A} = \frac{ne}{\epsilon_0} d$$

$$m \ddot{d} = -\frac{ne^2}{\epsilon_0} d$$



where d is thickness and also displacement.

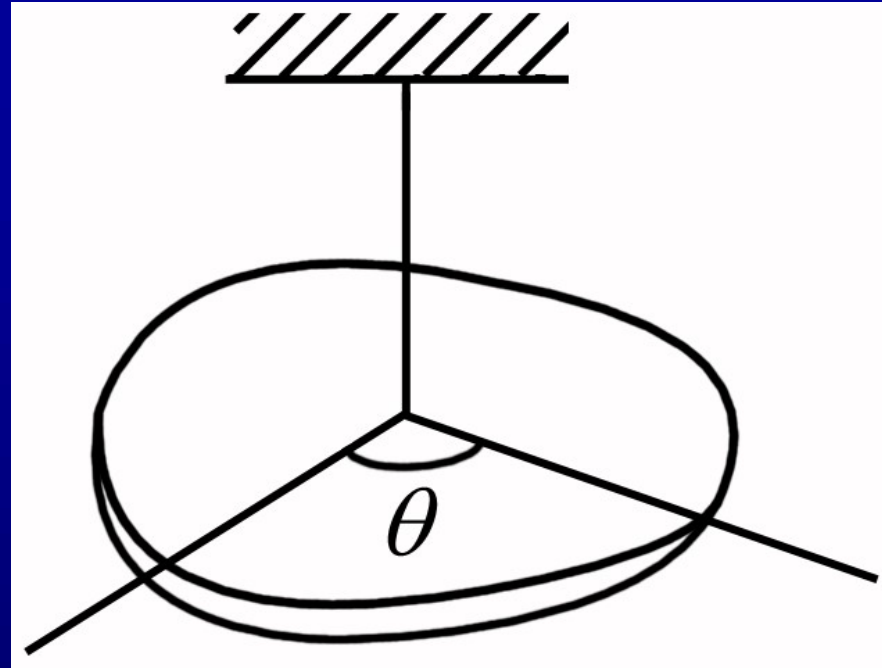
Torsion pendulum

$$M = -\kappa\theta$$

$$M = I\ddot{\theta}$$

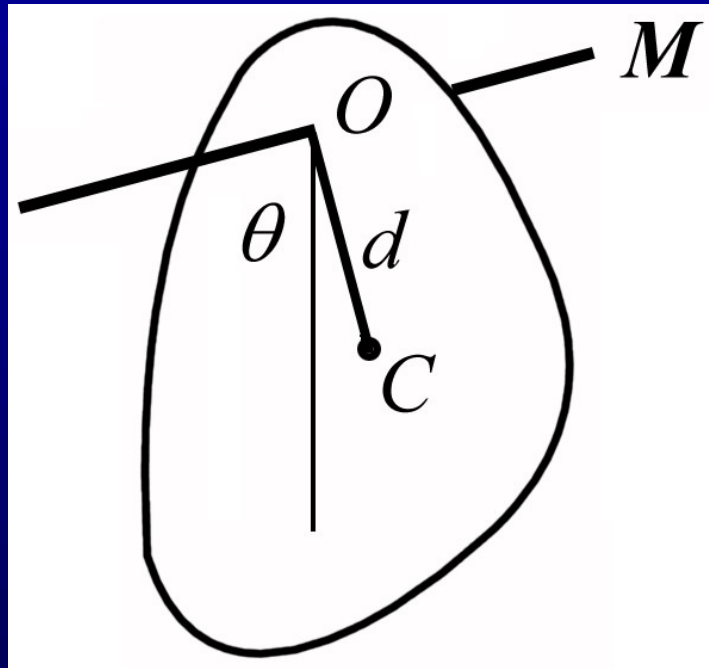
$$-\kappa\theta = I\ddot{\theta}$$

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$



Physical pendulum

(Compound pendulum)



$$M_{\text{ext}} = -dmg \sin \theta$$

$$I = I_C + md^2$$

$$M = I\ddot{\theta}$$
$$\ddot{\theta} + \frac{d}{I}mg \sin \theta = 0$$

For small oscillation

$$T = 2\pi \sqrt{\frac{I}{mgd}} \equiv 2\pi \sqrt{\frac{\ell_{\text{eff}}}{g}}$$

$$\ell_{\text{eff}} = \frac{I}{md} = d + \frac{I_C}{md}$$
$$= d + \frac{R_0^2}{d} > d$$

Displacement oscillator

+ mg or eE

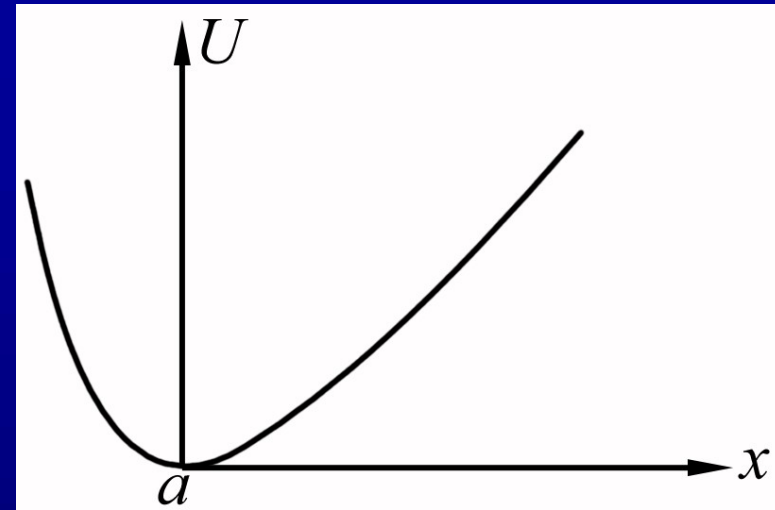
$$m\ddot{x} = -kx + F$$

$$m \frac{d^2}{dt^2} \left(x - \frac{F}{k} \right) = -k \left(x - \frac{F}{k} \right)$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k (x - x_0)^2 = E, \quad x_0 \equiv \frac{F}{k}$$

anharmonic oscillator

非谐

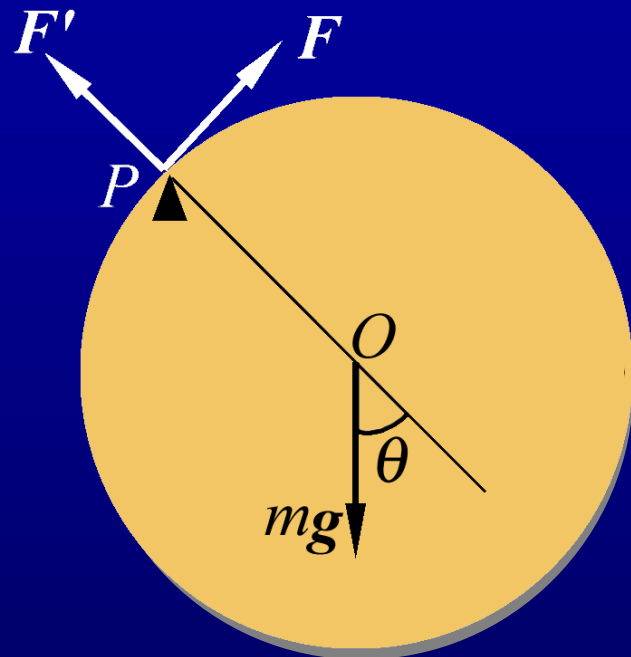


$$U(x) = U(a) + U'(a)(x - a) + \frac{1}{2!}U''(a)(x - a)^2 + \dots$$

$$F = -U''(a)(x - a)$$

$$\omega^2 = U''(a)/m$$

Torque equation is preferred



Disc pendulum

$$-Rmg \sin \theta = \left(\frac{1}{2}mR^2 + mR^2 \right) \ddot{\theta}$$

$$\xrightarrow{?} mg = m \frac{d^2}{dt^2} r_c$$

$$\xrightarrow{\quad} -mg \sin \theta = mR \ddot{\theta}$$

With respect to O

$$-RF = \frac{1}{2}mR^2 \ddot{\theta} \quad \text{or} \quad F = -\frac{1}{2}mR \ddot{\theta}$$

$$e_\rho: \quad mg \cos \theta - F_T = m(-L\dot{\theta}^2)$$

$$e_\theta: \quad -mg \sin \theta = mL\ddot{\theta}$$

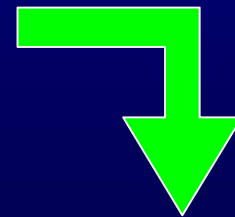
$$m \frac{d^2}{dt^2} r_C = mg + F' + F$$

$$mg \cos \theta - F' = m(-R\dot{\theta}^2)$$

$$-mg \sin \theta + F = mR\ddot{\theta} \quad \left. \vphantom{-mg \sin \theta + F = mR\ddot{\theta}} \right\}$$

$$-RF = \frac{1}{2}mR^2\ddot{\theta}$$

$$-Rmg \sin \theta = \left(\frac{1}{2}mR^2 + mR^2 \right) \ddot{\theta}$$



$$\ddot{\theta} + \frac{2g}{3R} \sin \theta = 0$$

$$\frac{1}{2} \dot{\theta}^2 = \frac{2g}{3R} (\cos \theta - \cos \theta_0)$$

$$F' = mg \cos \theta + mR \dot{\theta}^2$$

$$= mg \cos \theta + \frac{4}{3} mg (\cos \theta - \cos \theta_0)$$

$$= \frac{1}{3} mg (7 \cos \theta - 4 \cos \theta_0)$$

Simple pendulum

$$U(\theta) = mgl(1 - \cos \theta)$$

$$U(\theta_0) = \frac{1}{2} m \ell^2 \dot{\theta}^2 + U(\theta)$$

$$\dot{\theta}^2 + \frac{2g}{\ell} (\cos \theta_0 - \cos \theta) = 0$$

$$* T = 4 \sqrt{\frac{\ell}{g}} K\left(\sin \frac{\theta_0}{2}\right) \quad (\text{appendix B})$$

$$= 2\pi \sqrt{\frac{\ell}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{\theta_0}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4 \frac{\theta_0}{2} + \dots \right]$$

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n} n!^2} \right]^2 k^{2n}$$

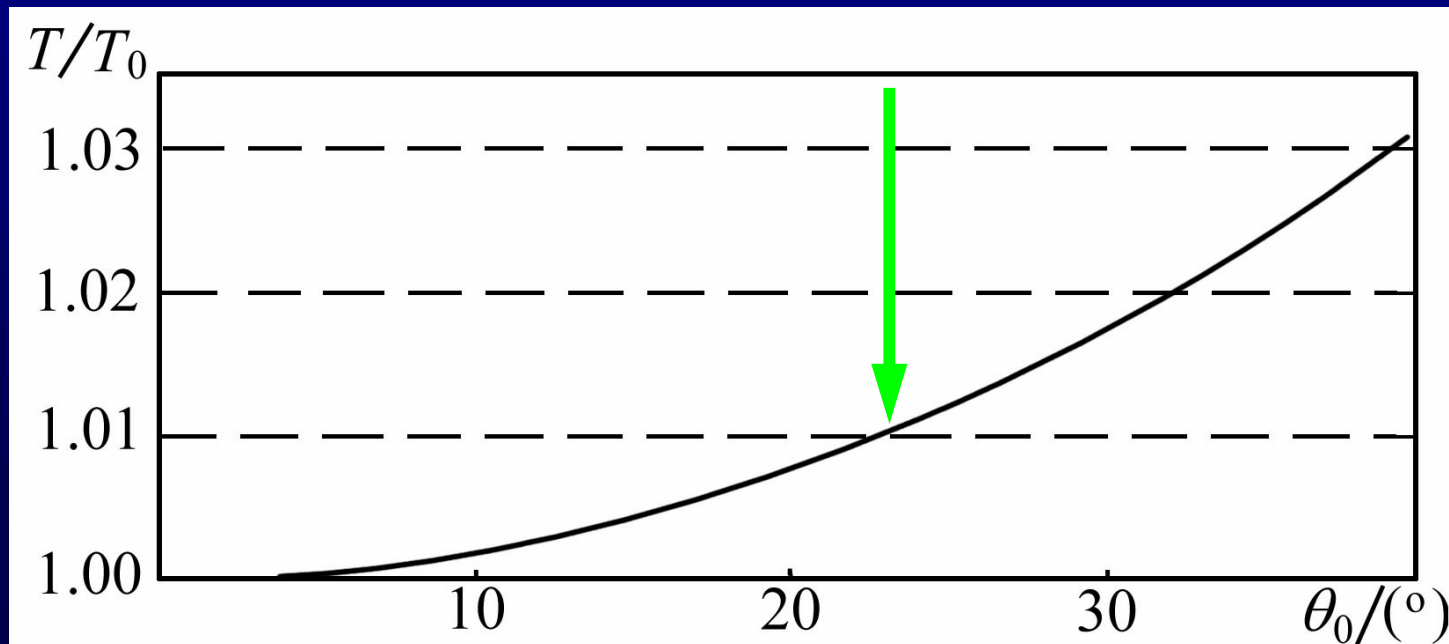
complete elliptic integral of the first kind

can not be expressed in terms of elementary functions

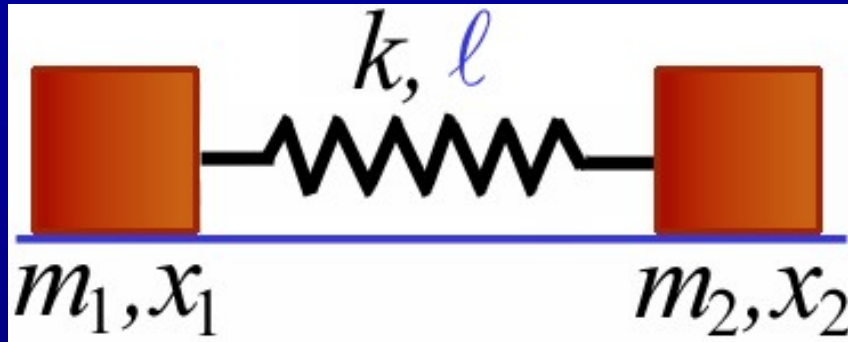
For small angle approximation

$$T \approx 2\pi\sqrt{\frac{\ell}{g}} \left(1 + \frac{1}{16}\theta_0^2\right) \equiv T_0 \left(1 + \frac{1}{16}\theta_0^2\right)$$

$$\frac{T}{T_0} - 1 \sim 1\% \quad \theta_0 = 0.4 \sim 23^\circ$$



7.2 Coupled oscillation



Two-body system:
reducible

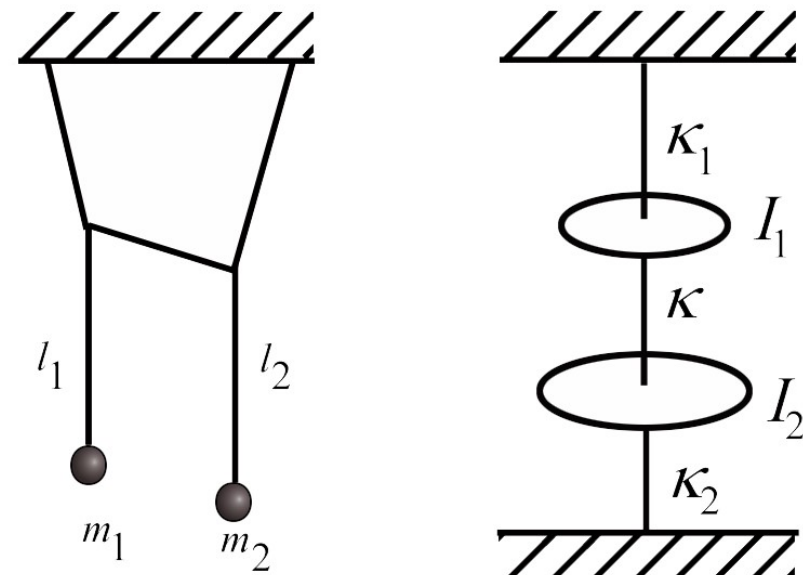
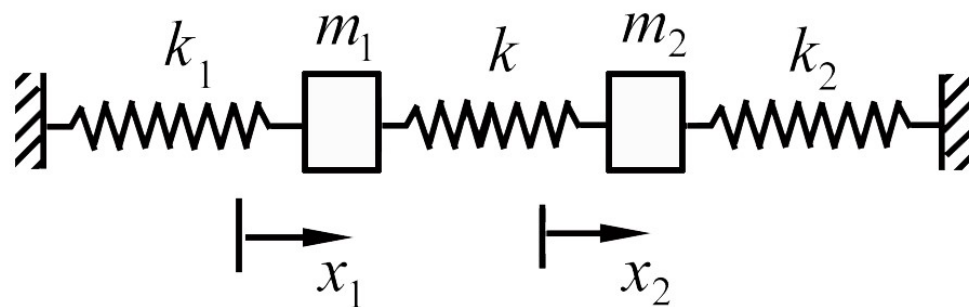
$$m_1 \ddot{x}_1 = k(x_2 - x_1 - \ell)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1 - \ell)$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad x = x_2 - x_1 - \ell$$

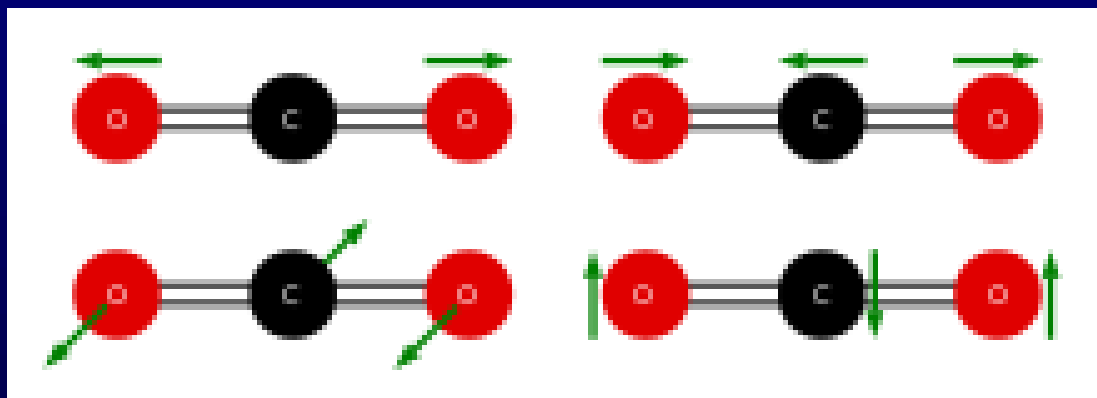
$$m_1 \ddot{x}_2 + m_2 \ddot{x}_2 = 0, \quad \mu \ddot{x} + k x = 0$$

A single
frequency



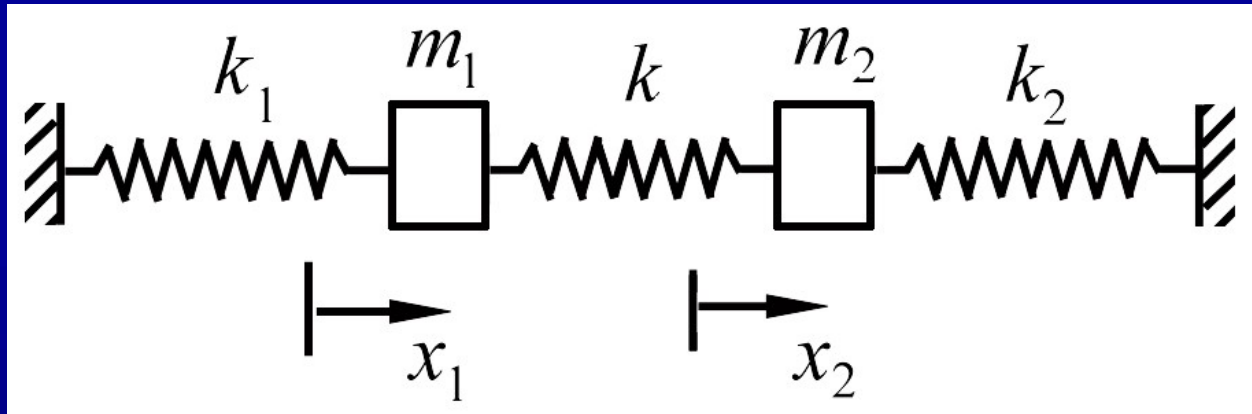
CO₂

normal modes:



$$\hbar\omega/k_B = 3360\text{K}, 1890\text{K}, 954\text{K}, 954\text{K}$$

Normal procedure



$$\begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 + k(x_2 - x_1) \\ m_2 \ddot{x}_2 = -k_2 x_2 - k(x_2 - x_1) \end{cases}$$

$$\begin{cases} \left(m_1 \frac{d^2}{dt^2} + k + k_1 \right) x_1 - k x_2 = 0 \\ k x_1 - \left(m_2 \frac{d^2}{dt^2} + k + k_2 \right) x_2 = 0 \end{cases}$$

$$x_1 = A_1 e^{i\omega t}, \quad x_2 = A_2 e^{i\omega t}$$

$$\begin{cases} \left(\omega^2 - \frac{k + k_1}{m_1} \right) A_1 + \frac{k}{m_1} A_2 = 0 \\ \frac{k}{m_2} A_1 + \left(\omega^2 - \frac{k + k_2}{m_2} \right) A_2 = 0 \end{cases}$$

Nontrivial solution requires $\Delta = 0$

$$x_i = \sum_{j=1}^4 a_{ij} \cos(\omega_j t + \varphi_{ij})$$

Potential energy consideration

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k_2x_2^2$$

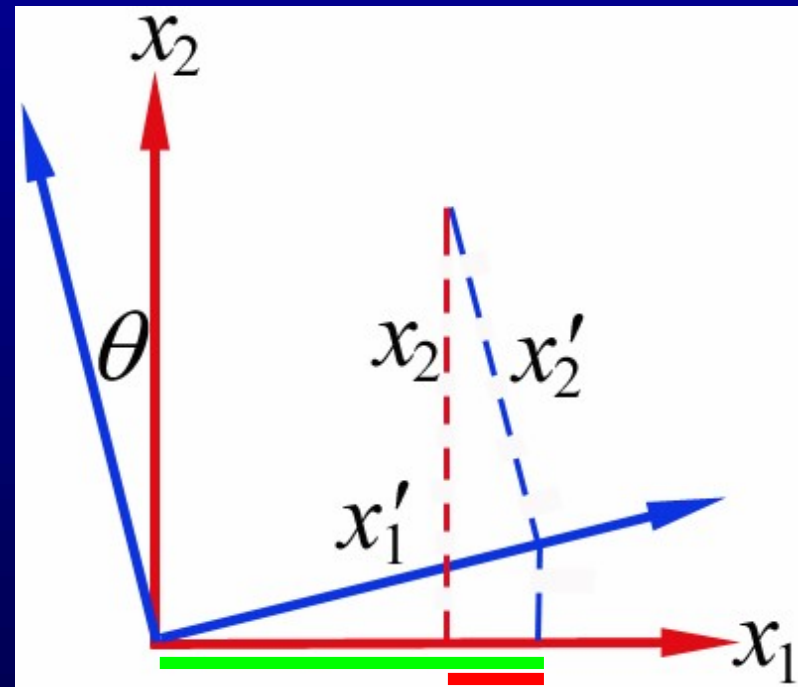
$$x_1 = \underline{x'_1 \cos \theta} - \underline{x'_2 \sin \theta}$$

$$x_2 = x'_1 \sin \theta + x'_2 \cos \theta$$

the coefficient of $x'_1 x'_2 = 0$

$$\cot 2\theta = \frac{k_2 - k_1}{2k}$$

$$U = \frac{1}{2}m_1\omega_1^2 x'^2_1 + \frac{1}{2}m_2\omega_2^2 x'^2_2$$



$$U = \frac{1}{2}m_1\omega_1^2 x_1'^2 + \frac{1}{2}m_2\omega_2^2 x_2'^2$$

$$m_1\omega_1^2 \equiv \frac{1}{2}(k_1 + k_2 + 2k) - \frac{1}{2}\sqrt{(k_1 - k_2)^2 + 4k^2}$$

$$m_2\omega_2^2 \equiv \frac{1}{2}(k_1 + k_2 + 2k) + \frac{1}{2}\sqrt{(k_1 - k_2)^2 + 4k^2}$$

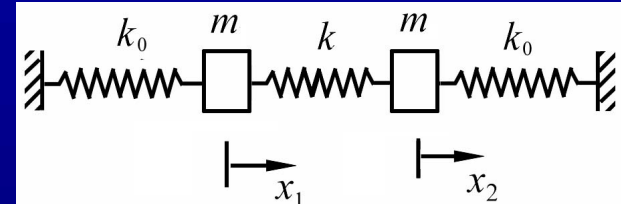
Simpler case $k_1 = k_2 = k_0; \quad m_1 = m_2 = m$

$$\begin{cases} m\ddot{x}_1 = -k_0x_1 + k(x_2 - x_1) \\ m\ddot{x}_2 = -k_0x_2 - k(x_2 - x_1) \end{cases}$$

$$\begin{cases} 2m \frac{d^2}{dt^2} \frac{(x_1 + x_2)}{2} = -2k_0 \frac{(x_1 + x_2)}{2} \\ \frac{m}{2} \frac{d^2}{dt^2} (x_1 - x_2) = -\left(\frac{k_0}{2} + k\right)(x_1 - x_2) \end{cases}$$

$$y_1 = \frac{x_1 + x_2}{2}, \quad y_2 = x_1 - x_2$$

$$\begin{cases} 2m \frac{d^2}{dt^2} y_1 = -2k_0 y_1 \\ \frac{m}{2} \frac{d^2}{dt^2} y_2 = -\left(\frac{k_0}{2} + k\right) y_2 \end{cases}$$



$$\omega_1 = \sqrt{k_0/m}, \quad \omega_2 = \sqrt{(k_0 + 2k)/m}$$

ω_1, ω_2 **normal frequencies** (简正频率)

y_1, y_2 **normal coordinates** (简正坐标)

normal modes (简正模式)

$$k = 6\pi/6a \quad \lambda = 2.00a \quad \omega_k = 2.00\omega$$



$$k = 5\pi/6a \quad \lambda = 2.40a \quad \omega_k = 1.93\omega$$



$$k = 4\pi/6a \quad \lambda = 3.00a \quad \omega_k = 1.73\omega$$



$$k = 3\pi/6a \quad \lambda = 4.00a \quad \omega_k = 1.41\omega$$



$$k = 2\pi/6a \quad \lambda = 6.00a \quad \omega_k = 1.00\omega$$



$$k = 1\pi/6a \quad \lambda = 12.00a \quad \omega_k = 0.52\omega$$

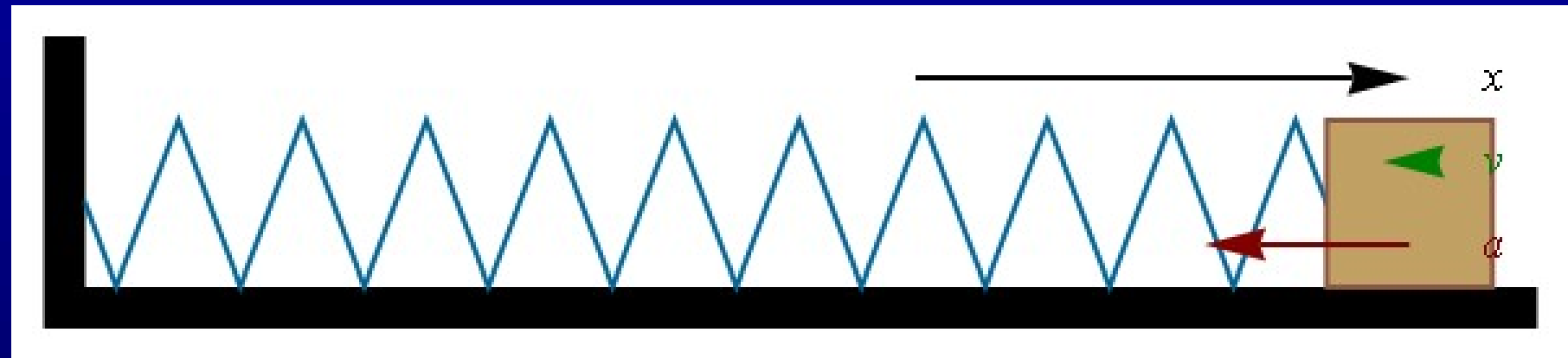


Normal modes of vibration progression through a 1D crystal.

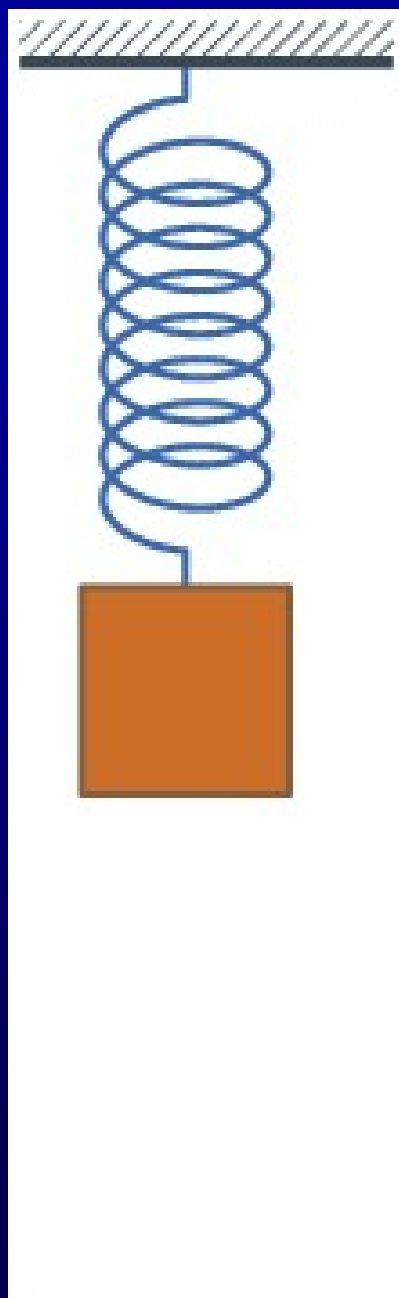
A phonon is a quantized normal mode of vibration occurring in a crystal lattice (lattice wave). It is a quasi-particle or elementary excitation of the many-body system.

Quasi-particle or elementary excitation is a very important concept in condensed matter physics.

Assignment: 7.1, 7.3/7.4, *7.5



The velocity and acceleration oscillate with a quarter and half a period delay



6.3 Damped Harmonic Motion

$$F = -\eta v$$

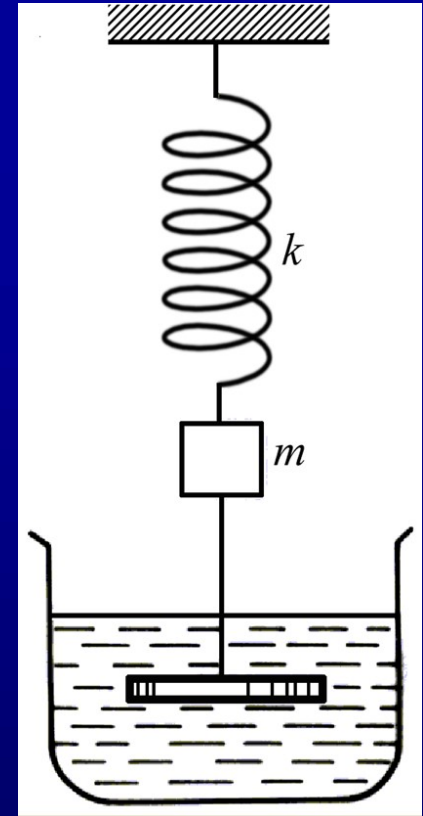
(for example, $F = 6\pi R \eta v$)

Equation of motion is

$$m \ddot{x} = -k x - \eta \dot{x}$$

or

$$\ddot{x} + \frac{\eta}{m} \dot{x} + \omega_0^2 x = 0$$



For equation $m \ddot{x} = -\eta \dot{x}$ (only damping)

$$m \frac{d\dot{x}}{dt} = -\eta \dot{x}$$

Scheme 1

$$\int \frac{d\dot{x}}{\dot{x}} = -\frac{\eta}{m} \int dt$$

$$\ln \dot{x} = -\frac{\eta}{m} t + C$$

$$\dot{x} = e^C e^{-\frac{\eta}{m} t}$$

$$\dot{x} = e^C e^{-\frac{\eta}{m}t}$$

$$\int dx = e^C \int e^{-\frac{\eta}{m}t} dt$$

$$x = -\frac{m}{\eta} e^C e^{-\frac{\eta}{m}t} + C'$$

$$\sim e^{-\frac{\eta}{m}t}$$

$$m \frac{d\dot{x}}{dt} = -\eta \dot{x}$$

Scheme 2

What function after derivation is itself (apart from a numerical factor)?

$$\dot{x} \sim e^{-\frac{\eta}{m}t}$$

What function after derivation is an exponent (apart from a numerical factor)?

$$x \sim e^{-\frac{\eta}{m}t}$$

A trial solution

$$x = Ae^{-\gamma t} \cos(\omega t + \varphi)$$


$$\dot{x} = Ae^{-\gamma t} [-\gamma \cos(\omega t + \varphi) - \omega \sin(\omega t + \varphi)]$$

$$\ddot{x} = Ae^{-\gamma t} [\gamma^2 \cos(\omega t + \varphi) + 2\gamma\omega \sin(\omega t + \varphi) - \omega^2 \cos(\omega t + \varphi)]$$

Substituting into equation of motion yields

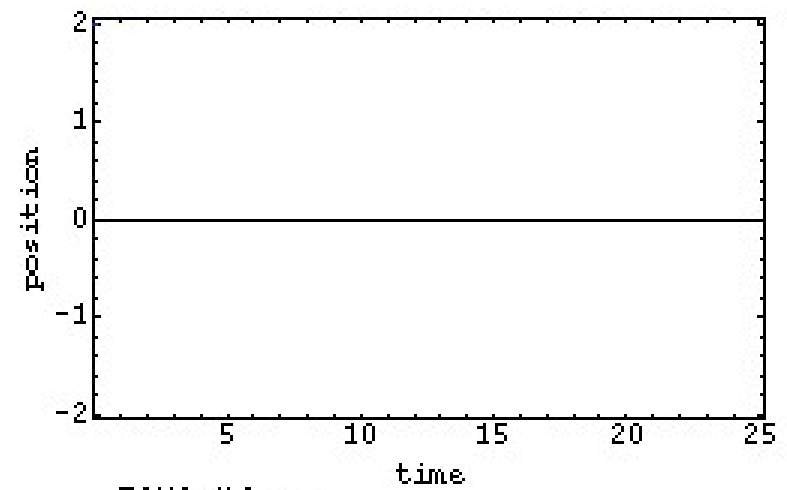
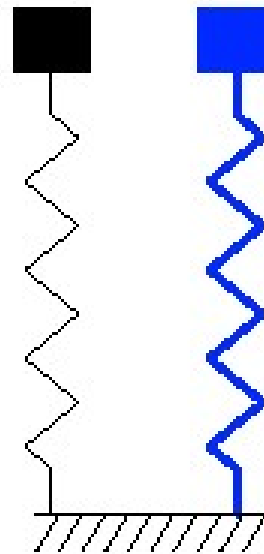
$$\left(\gamma^2 - \omega^2 - \boxed{\frac{\eta}{m}} \gamma + \omega_0^2 \right) \cos(\omega t + \varphi)$$

$$+ \left(2\gamma\omega - \boxed{\frac{\eta}{m}} \omega \right) \sin(\omega t + \varphi) = 0$$

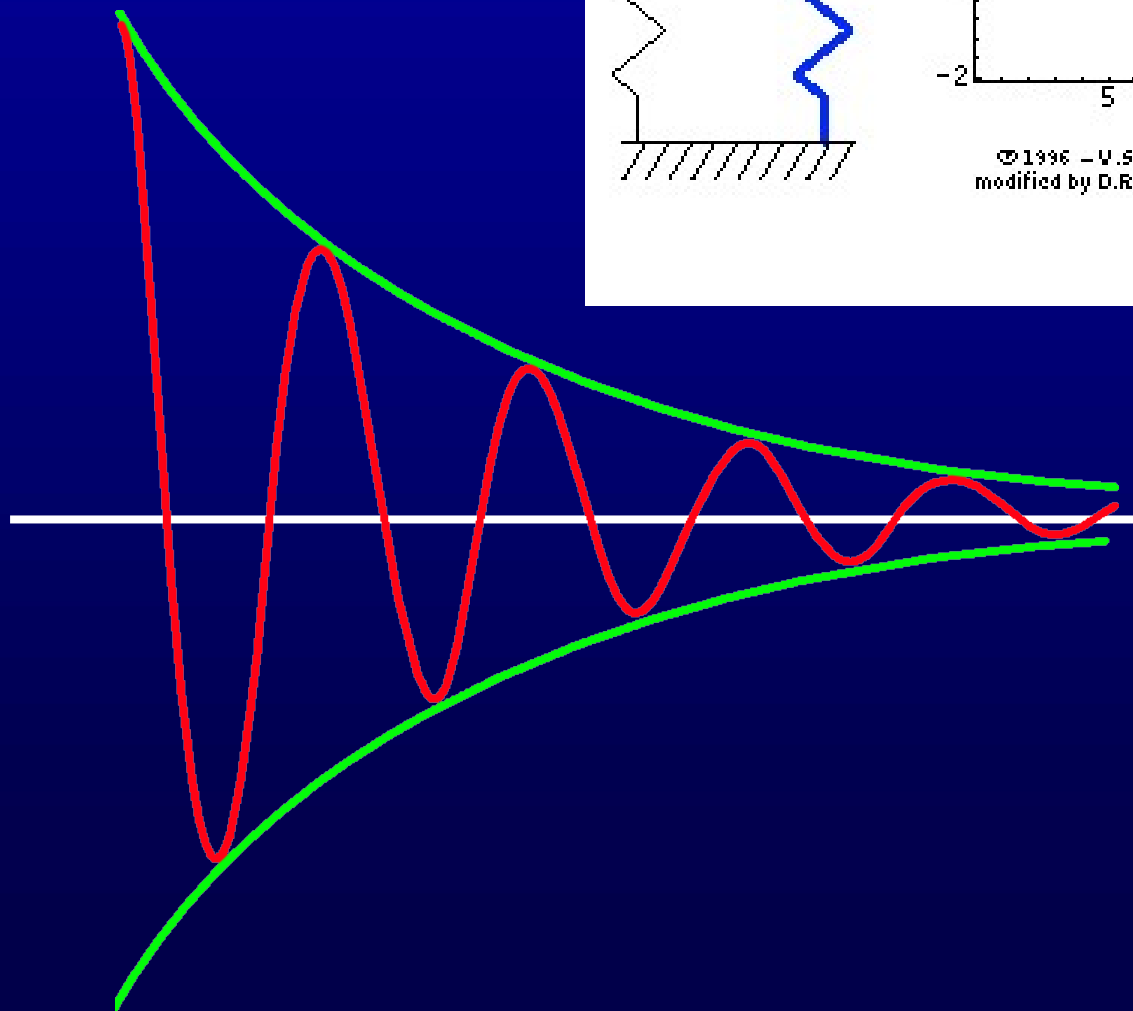
valid for arbitrary t

$$\gamma = \frac{\eta}{2m}, \quad \omega^2 = \omega_0^2 - \gamma^2$$

$$\gamma = \frac{\eta}{2m} < \omega_0$$



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modified by D.Russell, 1997



$$\gamma = \frac{\eta}{2m} < \omega_0$$

under damping
light damping

$$x = Ae^{-\gamma t} \cos(\omega t + \varphi), \quad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

$$\gamma = \frac{\eta}{2m} = \omega_0$$

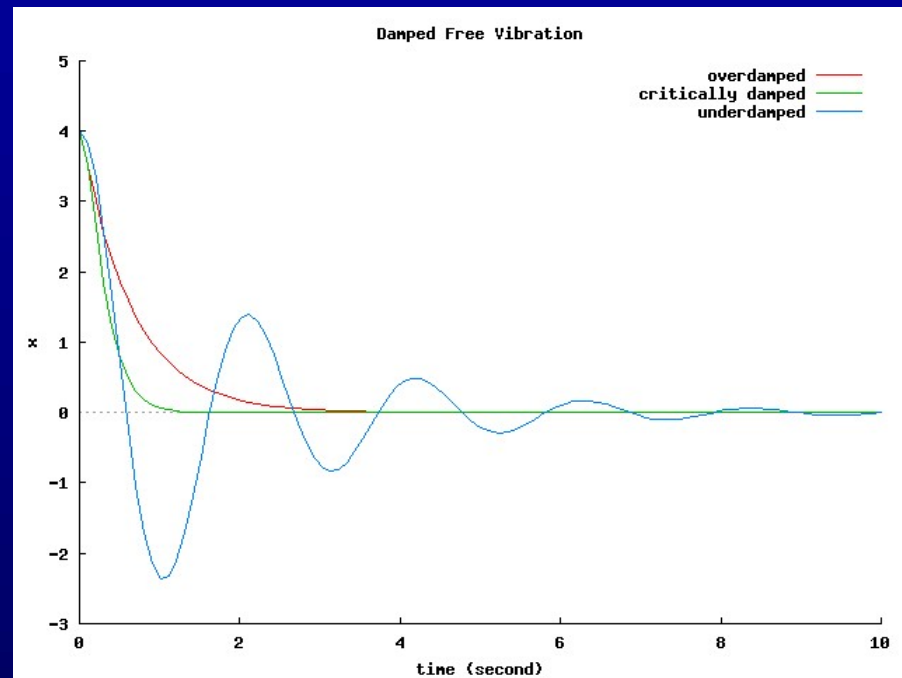
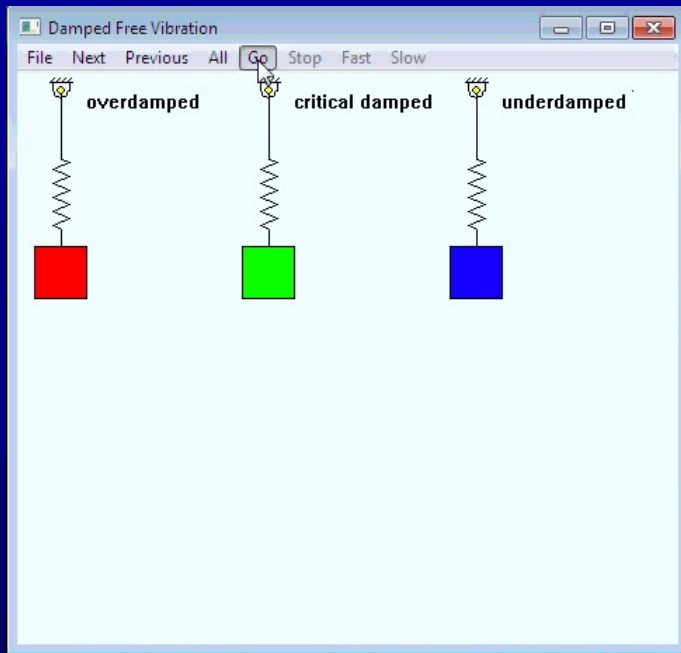
critical damping

$$x = (a + bt)e^{-\gamma t}$$

$$\gamma = \frac{\eta}{2m} > \omega_0$$

over damping
heavy damping

$$x = ae^{-\gamma t} [\cosh(\beta t) + b \sinh(\beta t)], \quad \beta = \sqrt{\gamma^2 - \omega_0^2}$$



The nth order linear differential equation: non-homogenous term

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-2}(x)y'' + a_{n-1}(x)y' + a_n(x)y(x) = b(x). \quad (1)$$

Introducing the linear differential operator

$$\hat{L} = \frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-2}(x)\frac{d^2}{dx^2} + a_{n-1}(x)\frac{d}{dx} + a_n(x), \quad (2)$$

the equation can be written as

$$\hat{L}y(x) = b(x). \quad (3)$$

The corresponding homogenous equation is $\hat{L}y(x) = 0$.

The nth order homogenous linear differential equation with constant coefficients is

$$y^{(n)} + a_1y^{(n-1)} + \dots + a_{n-2}y'' + a_{n-1}y' + a_ny(x) = 0. \quad (4)$$

where $a_i = \text{Const.}$ $i = 1, 2, \dots, n$.

Now , we search the particular solutions of Eq.(4) of form e^{zx} .

Substituting $y(x) = e^{zx}$ into (4), we have

$$\left(z^n + a_1 z^{n-1} + \dots + a_{n-2} z^2 + a_{n-1} z + a_n\right) e^{zx} = 0.$$

Factoring out e^{zx} which is never zero, we have that z satisfy the algebraic equation characteristic polynomial

$$F(z) = \underline{z^n + a_1 z^{n-1} + \dots + a_{n-2} z^2 + a_{n-1} z + a_n} = 0 \quad (5)$$

which is the characteristic equation of (4).

If Eq.(5) has n distinct roots z_i ($i = 1, 2, \dots, n$), then $e^{z_i x}$ ($i = 1, 2, \dots, n$) are n independent solutions of (4). Thus the general solution of (4) is

$$y(x) = \sum_{i=1}^n C_i e^{z_i x}. \quad (6)$$

If z_i is a root of (5) having multiplicity m , then we can show that

$$y(x) = x^k e^{z_i x}, \quad (1 \leq k \leq m-1, \quad k \in N) \quad (7)$$

is a solution of (4).

If z_i is a root of (5) having multiplicity m , then the characteristic polynomial can be factored as

$$F(z) = P(z)(z - z_i)^m.$$

Thus, the corresponding linear differential operator can also be factored as

$$\hat{L} = P\left(\frac{d}{dx}\right)\left(\frac{d}{dx} - z_i\right)^m.$$

Noting that $\frac{d}{dx}(x^k e^{z_i x}) = kx^{k-1} e^{z_i x} + x^k z_i e^{z_i x}$ or $\left(\frac{d}{dx} - z_i\right)(x^k e^{z_i x}) = kx^{k-1} e^{z_i x}$

obviously, we have

$$\left(\frac{d}{dx} - z_i\right)^k (x^k e^{z_i x}) = 0.$$

Therefore, by the fundamental theorem of algebra , there are also n independent particular solutions in the form of Eq. (7) or $e^{z_i x}$. And the general solution is the linear combination of them.

For the equation of damped harmonic motion

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0, \quad (8)$$

the characteristic equation is

$$z^2 + 2\gamma z + \omega_0^2 = 0.$$

The solution:

$$\gamma < \omega_0, \quad z_{1,2} = -\gamma \pm i\omega, \quad \omega = \sqrt{\omega_0^2 - \gamma^2};$$

$$\gamma = \omega_0, \quad z = -\gamma \quad \text{multiplicity} = 2;$$

$$\gamma > \omega_0, \quad z_{1,2} = -\gamma \pm \beta \quad \omega = \sqrt{\gamma^2 - \omega_0^2}.$$

Therefore, the general solution of (8) is:

$$\begin{aligned}\gamma < \omega_0, \quad x(t) &= C_1 e^{(-\gamma + i\omega)t} + C_2 e^{(-\gamma - i\omega)t} \\ &= e^{-\gamma t} \left[(C_1 + C_2) \cos \omega t + i(C_1 - C_2) \sin \omega t \right]\end{aligned}$$

$$\begin{aligned}\operatorname{Re}(x(t)) &= e^{-\gamma t} \left[\tilde{C}_1 \cos \omega t + \hat{C}_2 \sin \omega t \right] \\ &= A e^{-\gamma t} \cos(\omega t + \varphi);\end{aligned}$$

$$\gamma = \omega_0, \quad x(t) = (a + bt)e^{-\gamma t};$$

$$\begin{aligned}\gamma > \omega_0, \quad x(t) &= C_1 e^{-\gamma + \beta t} + C_2 e^{-\gamma - \beta t} = C_1 e^{-\gamma t} \left[e^{\beta t} + \frac{C_2}{C_1} e^{-\beta t} \right] \\ &= C_1 e^{-\gamma t} \left\{ [\sinh(\beta t) + \cosh(\beta t)] + \frac{C_2}{C_1} [\cosh(\beta t) - \sinh(\beta t)] \right\} \\ &= a e^{-\gamma t} [\cosh(\beta t) + b \sinh(\beta t)].\end{aligned}$$

In simple harmonic oscillation

A, ω, φ **basic**

In coupled oscillation

$\omega_1, \omega_2, \dots$ **essential**

**though different A_i, φ_i may lead to
numerous solutions, but not essential.**

In damped oscillation

what is essential?

Energy flow

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt} \left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \right) \\ &= m \dot{x} \ddot{x} + k x \dot{x} = (m \ddot{x} + k x) \dot{x} \\ &= -\eta \dot{x}^2 < 0\end{aligned}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m A^2 \omega_0^2 e^{-2\gamma t} \left\{ 1 + \frac{\gamma}{\omega_0} \sin[2(\omega t + \varphi + \alpha)] \right\}$$

$$\propto A^2 \omega_0^2 e^{-2\gamma t}$$

Oscillating factor

$$\left(\tan 2\alpha \equiv \frac{\gamma}{\omega} \right)$$

$$\langle E \rangle \approx \frac{1}{2} m A^2 \omega_0^2 e^{-2\gamma t}$$

$$\ln\langle E \rangle = \ln\left(\frac{1}{2}mA^2\omega_0^2\right) + \ln e^{-2\gamma t}$$

$$= \ln\left(\frac{1}{2}mA^2\omega_0^2\right) - 2\gamma t$$

$$-\frac{1}{\langle E \rangle} \frac{d\langle E \rangle}{dt} = -\frac{d \ln\langle E \rangle}{dt} = 2\gamma$$

$$-\frac{1}{\langle E \rangle} \frac{d\langle E \rangle}{dt} T = 2\gamma \frac{2\pi}{\omega} \equiv \frac{2\pi}{Q}$$

Quality factor $Q \equiv \frac{\omega}{2\gamma} = \frac{m\omega}{\eta}$

In Tangshang, northern China, before April 1976 earthquake the average quality factor was $\overline{Q} = 666$ after $\overline{Q} = 180$. This means after earthquake, rocks were broken, earthquake waves decay increased.

Assignment 7.7

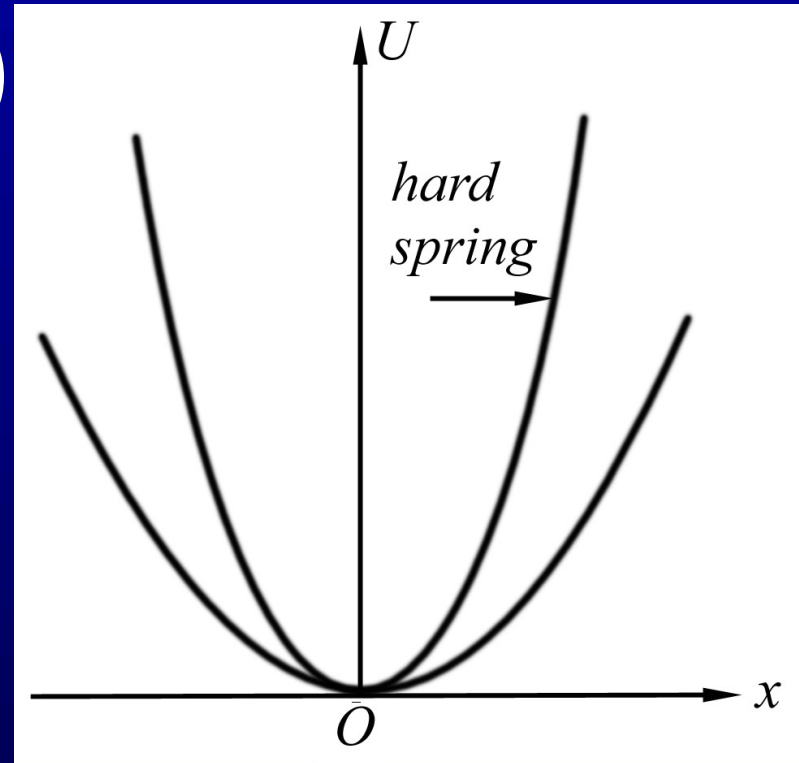
* 7.4 Nonlinear oscillation

$$F = -(1 + Bx^2)kx \quad (B > 0)$$

$$\ddot{x} + \alpha x + \beta x^3 = 0$$

$$\frac{U}{m} = -\int \frac{F}{m} dx = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4$$

Hard potential



$$F = -\left(1 - Bx^2\right)kx \quad (B > 0)$$

$$\frac{U}{m} = \frac{1}{2}\alpha x^2 - \frac{1}{4}\beta' x^4$$

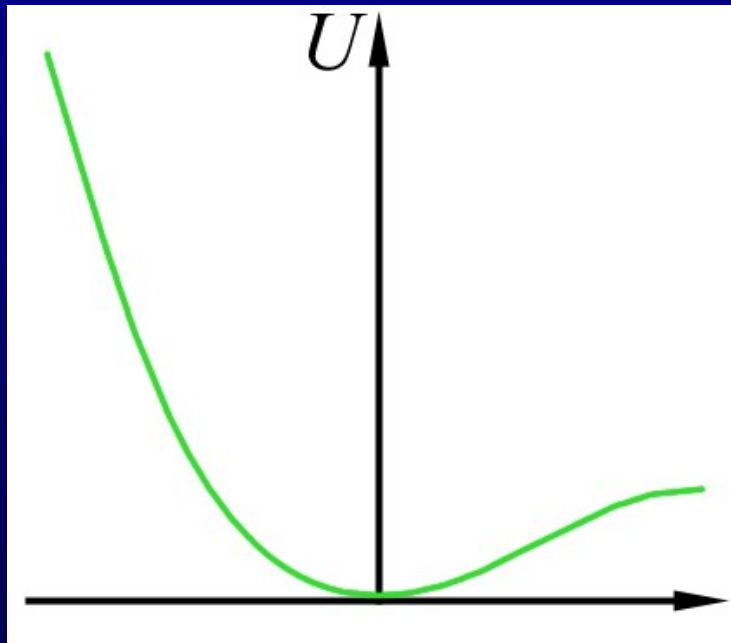
Soft potential

In simple pendulum

$$\begin{aligned} -mg \sin \theta &\approx -mg \left(\theta - \frac{1}{3!} \theta^3 + \dots \right) \\ &= -mg \theta \left(1 - \frac{1}{3!} \theta^2 + \dots \right) \end{aligned} \quad B = \frac{1}{3!}$$

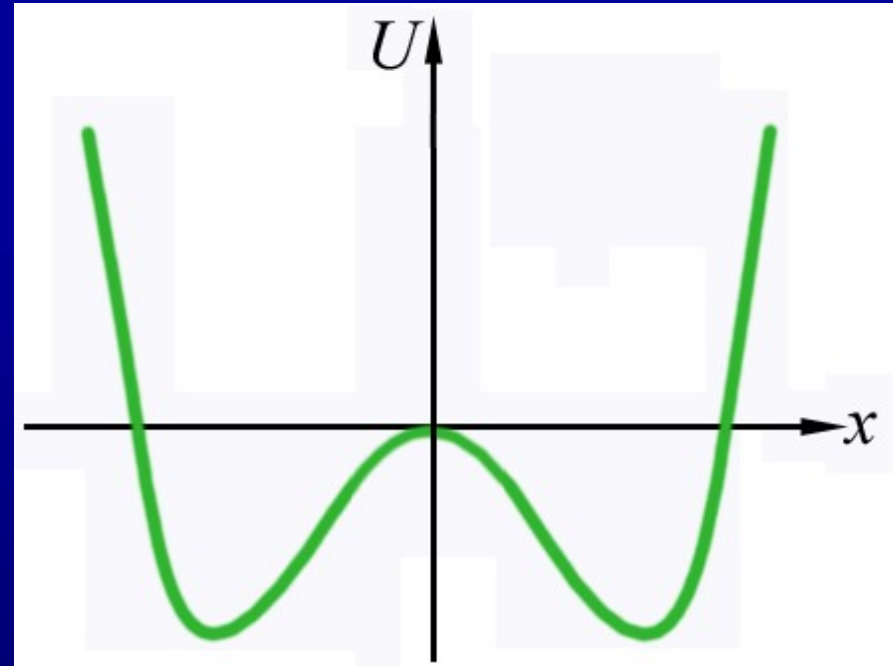
anharmonic oscillator

$$F = -(1 + Bx)kx \quad (B > 0)$$



Two-well potential

$$u = -\frac{1}{2}\alpha'x^2 + \frac{1}{4}\beta x^4$$



$\alpha = 0$ critical bifurcation

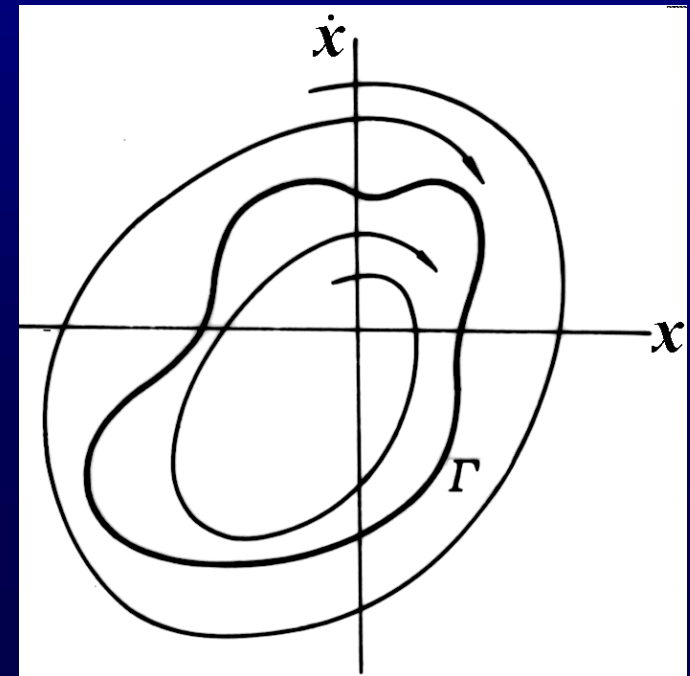
From this value to positive we have hard potential

$$u = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4$$

attractor

- equilibrium (point)
- periodic motion or **a limit cycle** (closed curve in phase plane)

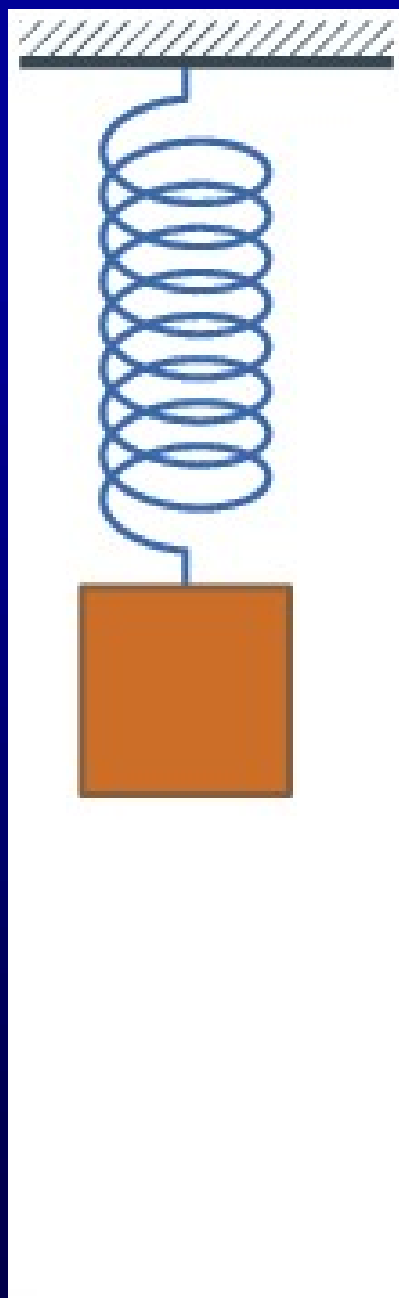
$$\ddot{\theta} + \alpha \sin \theta = T + \gamma \dot{\theta}(1 - \beta \dot{\theta}^2)$$

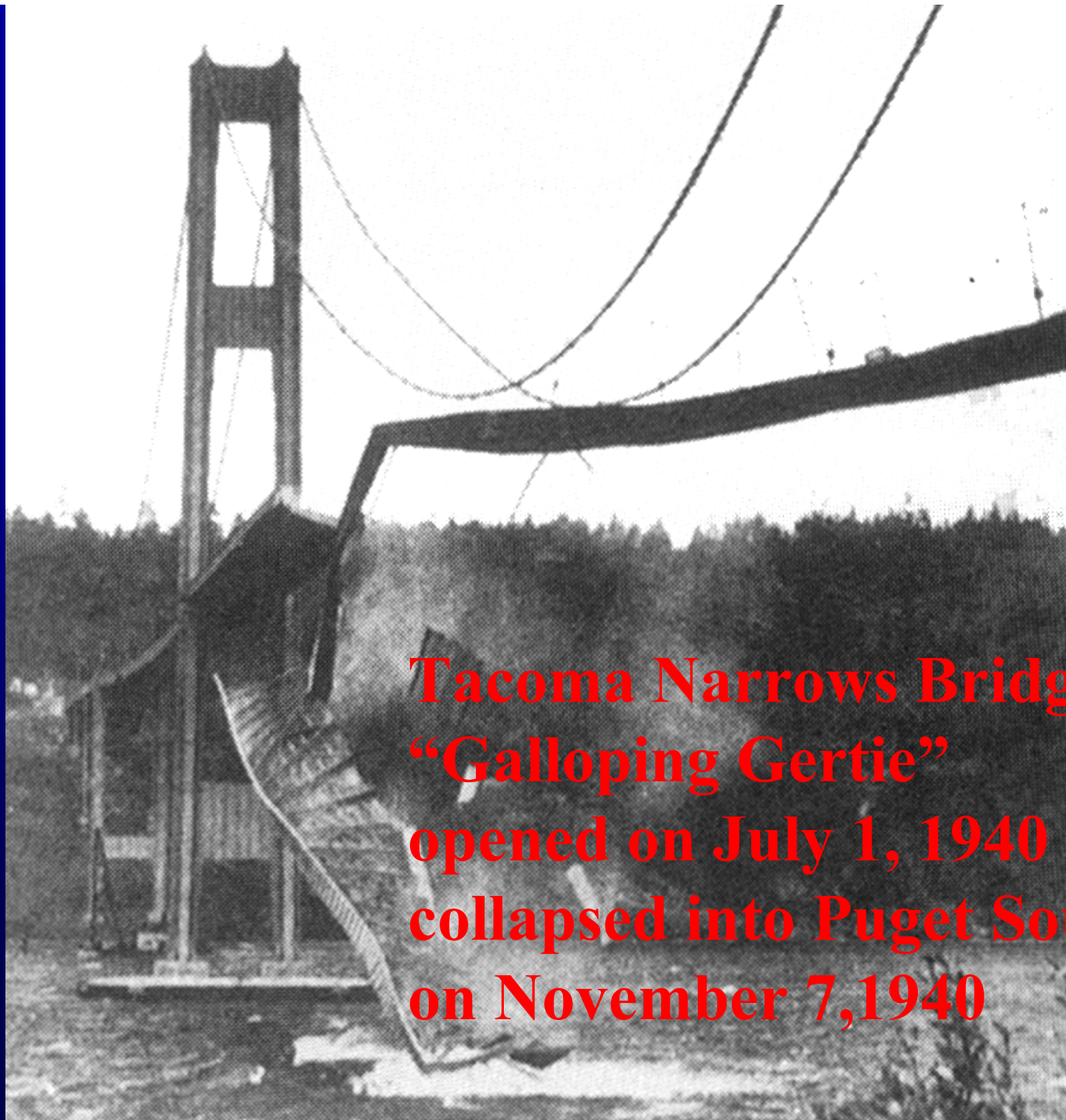


- quasi-periodic motion (surface in a 3D phase space)

$$f(t) = \cos(\omega t) + \cos(\pi \omega t)$$

A strange attractor $\sim$ fractal set





Tacoma Narrows Bridge
“Galloping Gertie”
opened on July 1, 1940
collapsed into Puget Sound
on November 7, 1940

7.5 Forced oscillation under friction

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

$$2\gamma = \frac{\eta}{m}$$

$$\omega_0^2 = \frac{k}{m}$$

$$F_0 \cos \omega t$$

ω_0 the (undamped) natural frequency

ω the driving frequency

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \quad (1)$$

Non-homogenous term

The equation is a second order non-homogenous linear ordinary differential equation with constant coefficients
二阶非齐次常系数线性常微分方程

The corresponding homogenous equation is

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0.$$

The general solution of (1) can be written as

$$x(t) = \underline{x_h(t)} + \underline{x_p(t)}$$

general solution of the corresponding
homogenous equation

a particular solution

ordinary differential equation 常微分方程 ODE

partial differential equation 偏微分方程 PDE

We expect a solution of following form:

$$x = x_d + x_F$$

where x_d is transient term due to damping.

Particular solution of the equation is

$$x_F = A \cos(\omega t + \varphi)$$

or

$$A \sin(\omega t - \alpha)$$

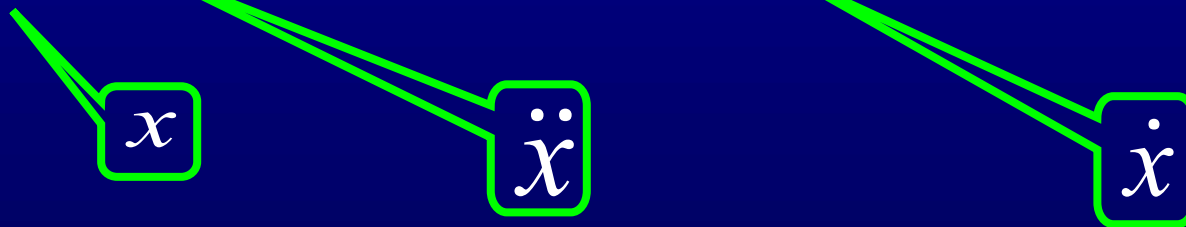
nothing but a math trick.

Trial solution

$$x = A \sin(\omega t - \alpha)$$

Substitute it into equation of motion

$$\left[(\omega_0^2 - \omega^2) \sin(\omega t - \alpha) + 2\gamma\omega \cos(\omega t - \alpha) \right] A = \frac{F_0}{m} \cos \omega t$$



$$(\omega_0^2 - \omega^2) \cos \alpha + 2\gamma\omega \sin \alpha = 0 \quad \sin \omega t$$

$$\left[2\gamma\omega \cos \alpha - (\omega_0^2 - \omega^2) \sin \alpha \right] A = \frac{F_0}{m} \cos \omega t$$

$$\tan \alpha = \frac{\omega^2 - \omega_0^2}{2\gamma\omega}$$

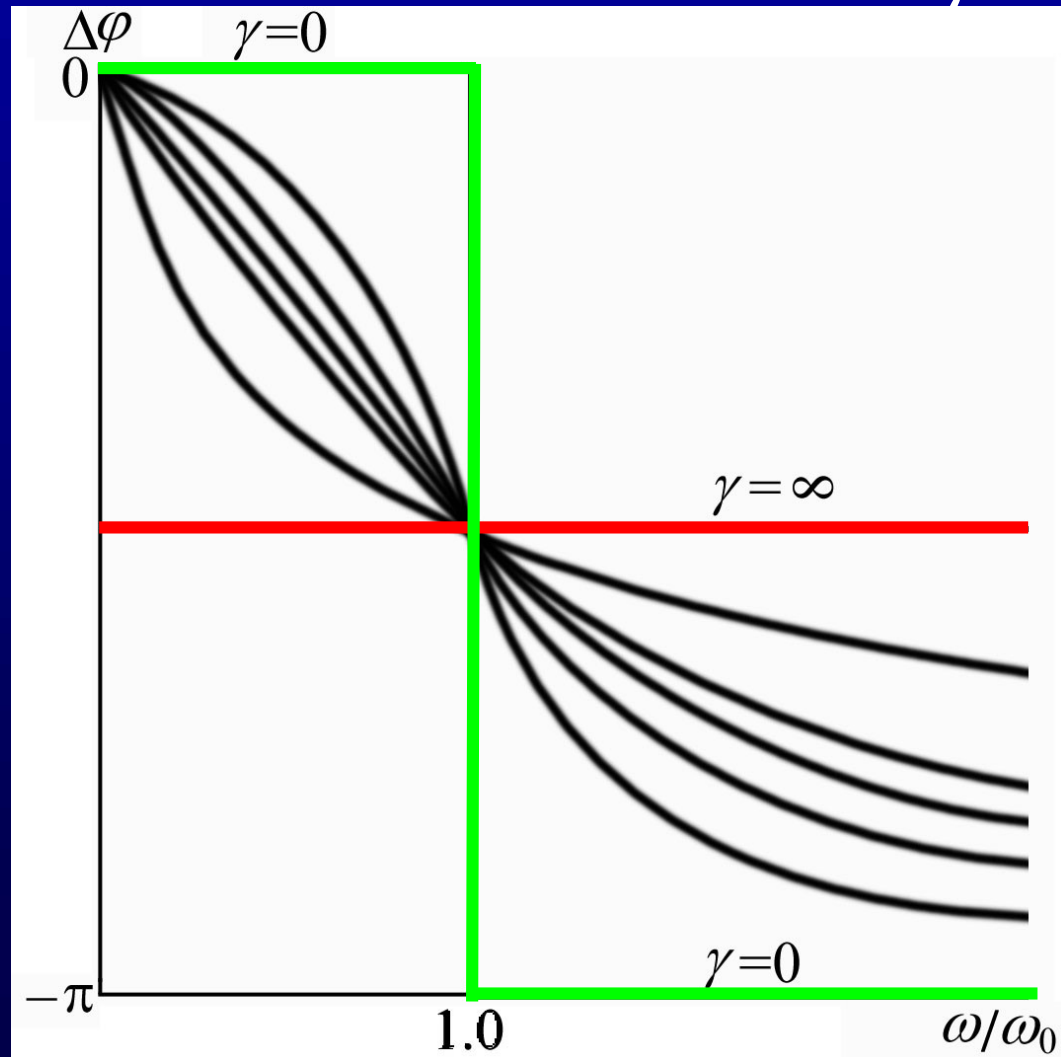
$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$

$$x = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}} \times$$

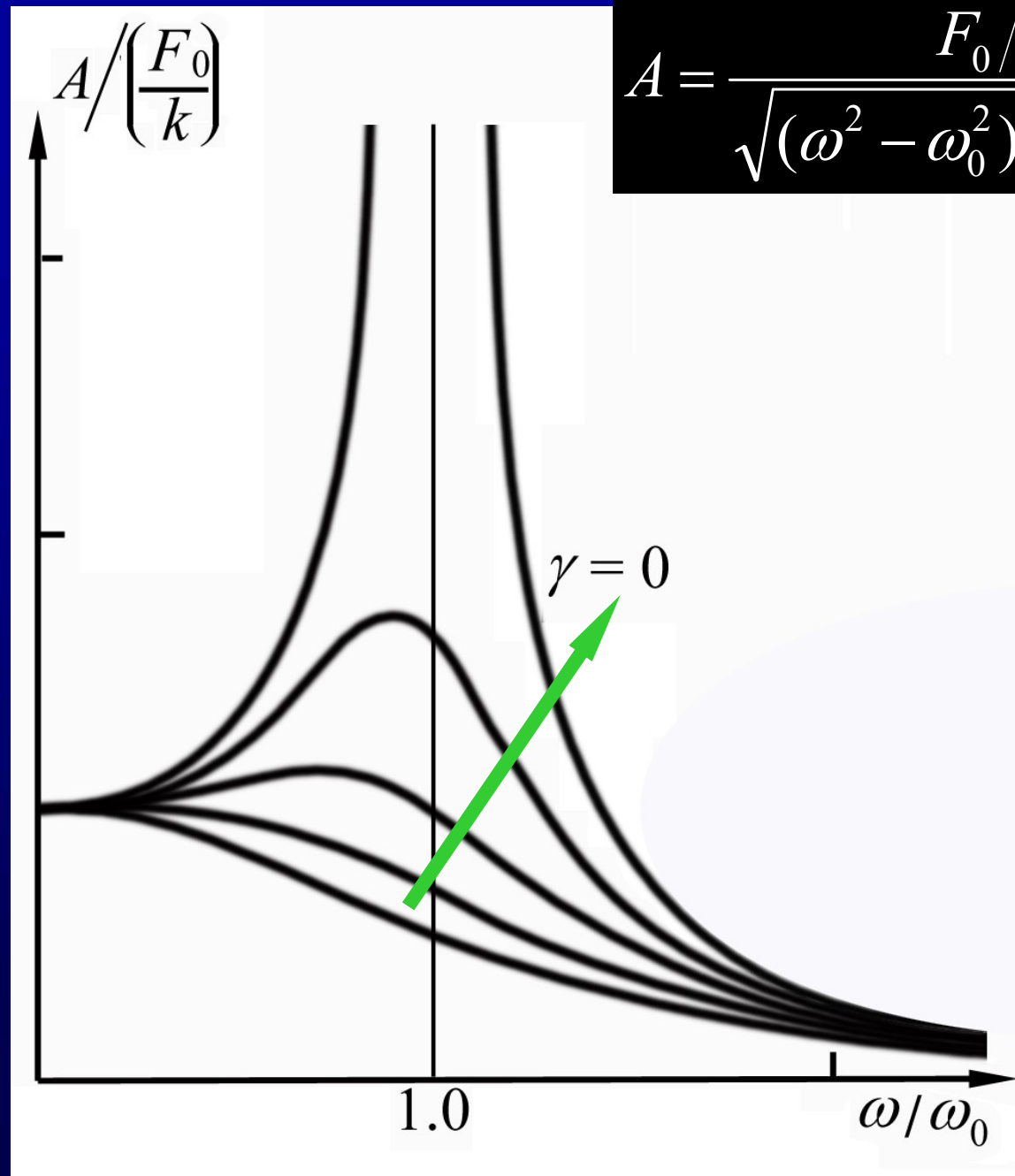
$$\cos \left(\omega t - \frac{\pi}{2} + \arctan \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \right)$$

$$F_0 \cos \omega t$$

$$\Delta\varphi = \varphi_x - \varphi_F = -\frac{\pi}{2} + \arctan \frac{\omega_0^2 - \omega^2}{2\gamma\omega}$$



$$\begin{aligned}\omega &= 0 \\ A &= \frac{F_0/m}{\omega_0^2} \\ &= \frac{F_0}{k}\end{aligned}$$



$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}}$$

To find extremum

$$\frac{\partial A}{\partial \omega} = 0 \quad \text{or} \quad \frac{\partial}{\partial \omega^2} \left[(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2 \right] = 0$$
$$2(\omega^2 - \omega_0^2) + 4\gamma^2 = 0$$

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$

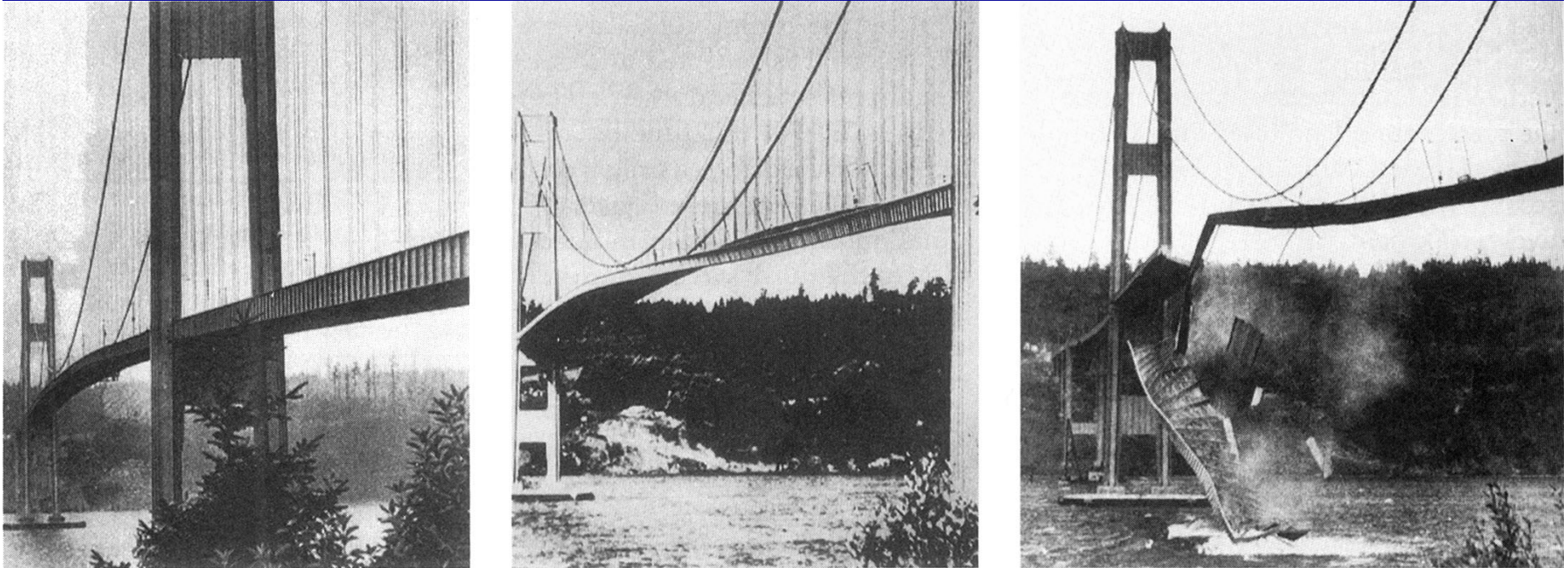
To find extremum

$$\frac{\partial A}{\partial \omega} = 0 \quad \text{or} \quad \frac{\partial}{\partial \omega^2} \left[(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2 \right] = 0$$

$$\omega^2 = \omega_0^2 - 2\gamma^2$$

$$x_m = \frac{F_0}{k} \frac{\omega_0^2}{2\gamma\sqrt{\omega_0^2 - \gamma^2}} \cos\left(\omega t - \frac{\pi}{2} + \arctan \frac{\gamma}{\omega}\right)$$

$\gamma \rightarrow 0 \quad x \rightarrow \infty$ **Displacement resonance**

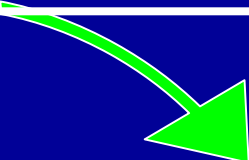


Turbulent winds set up standing waves in the Tacoma Narrows suspension bridge leading to its collapse on November 7, 1940, just four months after it had been opened for traffic.

The shape of resonance curve is important for sensor design.

For capacity sensor using displacement variation characteristics, one may let it work in the flat zone $\omega / \omega_0 \ll 1$.

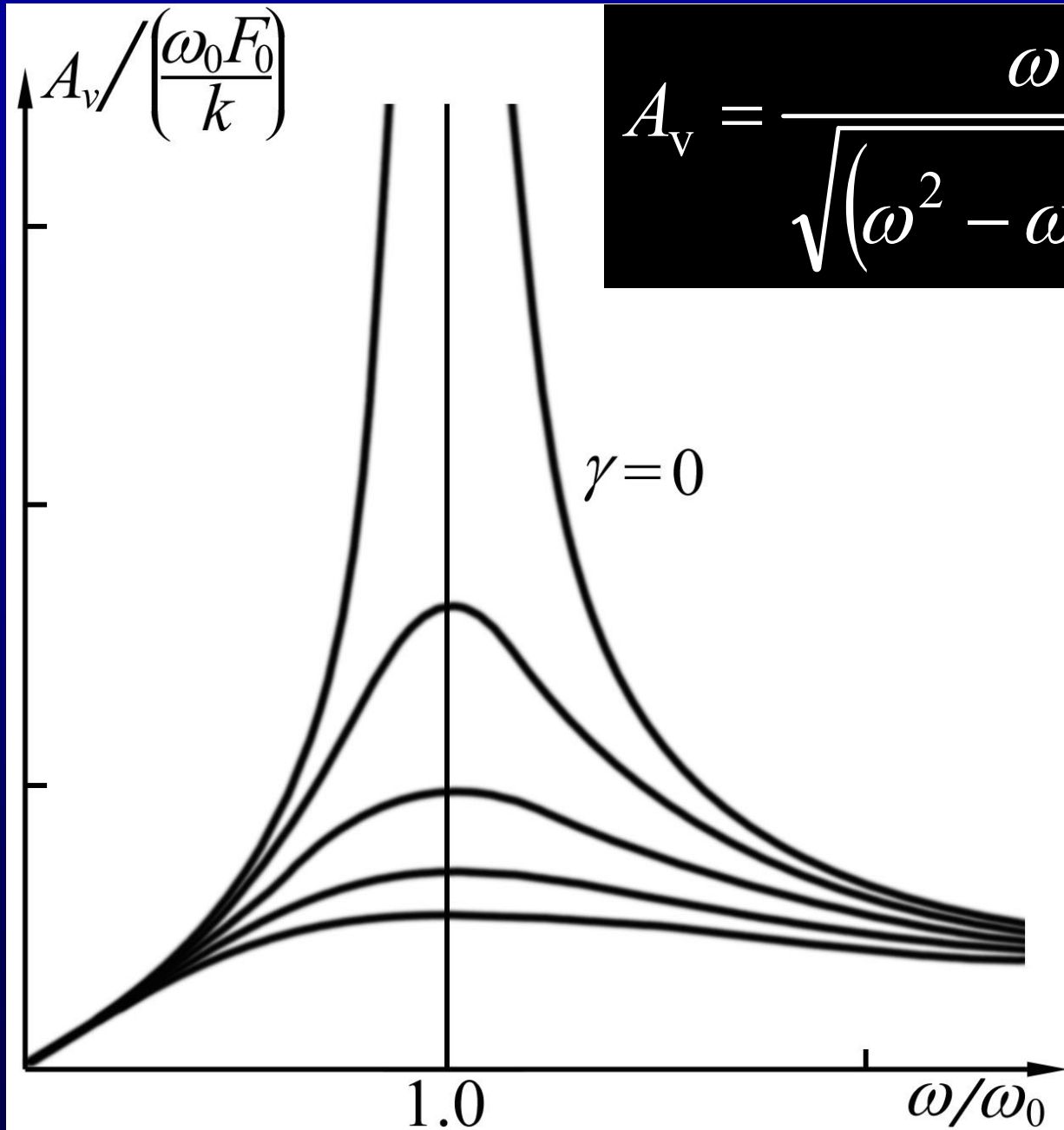
Then sensor has a flat frequency response.



$$\dot{x} = \frac{\omega F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}} \cos\left(\omega t + \arctan \frac{\omega_0^2 - \omega^2}{2\gamma\omega}\right)$$

The extremum appear at $\omega = \omega_0$ and is γ -independent. Then

$$\dot{x} = \frac{F_0}{k} \frac{\omega_0^2}{2\gamma} \cos(\omega t)$$



$$A_v = \frac{\omega F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}}$$

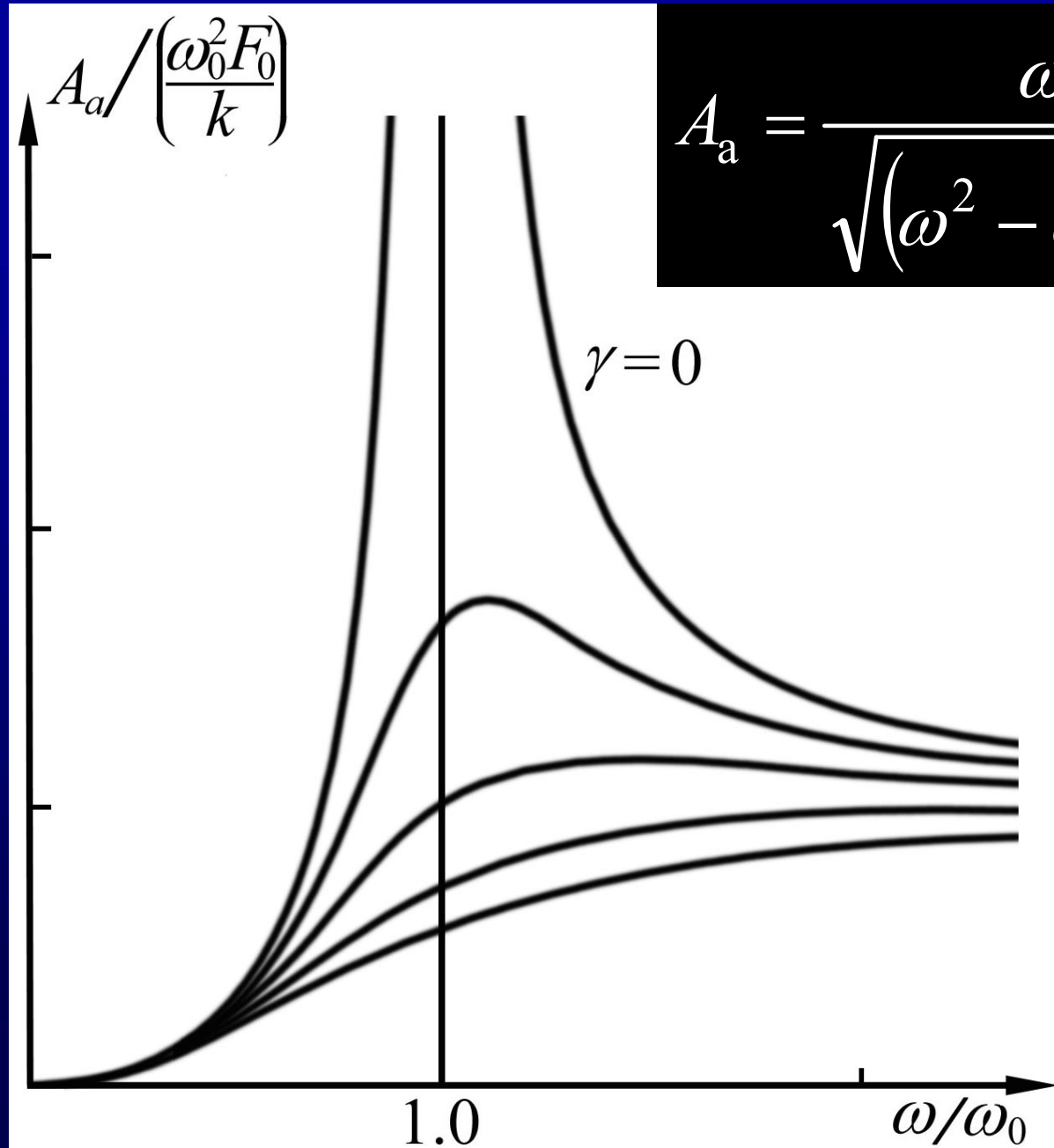
For the induced emf of coil-driven loudspeaker or microphone is proportional to the speed of its membrane.

Sensor working in resonance zone has better frequency response and higher sensitivity.

$$\ddot{x} = \frac{\omega^2 F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}} \cos \left(\omega t + \frac{\pi}{2} + \arctan \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \right)$$

The extremum appear at $\omega = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\gamma^2}}$

$$\ddot{x}_m = \frac{\omega_0^2 F_0}{k} \frac{\omega_0^2}{2\gamma \sqrt{\omega_0^2 - \gamma^2}} \cos \left(\omega t + \frac{\pi}{2} - \arctan \frac{\gamma}{\sqrt{\omega_0^2 - 2\gamma^2}} \right)$$



$$A_a = \frac{\omega^2 F_0 / m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}}$$

Acceleration resonance curve is flat at zone $\omega / \omega_0 \gg 1$.

But when using accelerometer to measure acceleration of the source, one has to consider the requirement of oscillation pickup, i.e., $\omega / \omega_a \ll 1$, where ω_a is the harmonic frequency of accelerometer.

Resonance phenomena have many applications in various fields. Some may be listed below:

- electronic circuits,**
- acoustics,**
- atomic process,**
- nuclear magnetic resonance,**
- Mössbauer effect,**
- elementary particle resonance etc.**

Resonance frequency of human body

“system”	Resonance frequency/Hz
chest-abdomen	3~6
head-neck-shoulder	20~30
eyeball	60~90
lower jaw-skull	100~200

Assignment:7.8

1999年1月4日 重庆綦江虹桥垮塌案

生死无常

惨剧发生前，一点儿预兆都没有。元旦刚过，傍晚时分，彩虹桥上一派祥和的气氛，许多人是带着高兴的心情踏上这座桥的。幸存者、武警驻綦江中队副班长刘爱兵躺在綦江县人民医院的病床上对记者说，中队每日训练跑过彩虹桥时，都有一种像汽车驶过桥梁时的共振感，但汽车驶过桥不会塌，人跑过桥怎么会塌呢？这天傍晚出营房时，战士们情绪饱满，口号响亮。因为刚刚休完假，人也齐，除了哨兵等人，能参加训练的24个人都出来了。中队驻地离桥头约1500米，跑到500米处，一名战士因牙疼打报告出列；另一名战士在跑到桥头台阶时鞋带开了。结果弯腰系鞋带的彭继军和牙疼的战友躲过了坍塌的一瞬。其余跑上桥的22名士兵18死3伤，只有1人被幸运地甩到江中，毫发未伤。

伤者中黎金海老汉71岁，年纪最大，他回忆说，吃完晚饭他照例到桥上散步，见到认识的方云秀带着儿子上桥，还跟她打了招呼，后来才知道母子俩都死了。他还看见两个小学生背着书包在桥上打闹，就呵斥一句：“放学了，还不回

家写作业！”他让手里牵着的狗猛叫，吓得两个孩子跑下桥去，没想到竟救了两个孩子的命。

黎金海老汉说，桥塌之前他正站在桥上听摆卦摊的刘德伦给他算命。事后两人双双住院，黎金海问刘德伦：“桥塌你怎么没算出来？”刘德伦还在嘴硬：“我当然算出来了，但桥塌应该在本月中旬，怎么跑到上旬来了呢？”

坍塌瞬间

1月7日上午，记者在綦江东岸追踪一辆灵车采访时，几十名群众围上来，情绪激动地诉说了桥塌时那令人惊心动魄的一刻。

綦江造纸厂职工张锡维把记者带到桥头，指着一路之隔的98洪火锅饭馆说，当时他正和几个朋友在那儿吃饭，刚要喝第二杯酒时，猛地听到“轰”的一声。他以为饭馆旁的高楼塌了，回身就往外跑，一抬头却看到彩虹桥塌了。西半段先垮，然后东半段垮，隐约看到桥上有人，一晃就随着断桥下去了。江里传出哭喊声。连接东桥头的那块桥板，像根孤木那样支着，20多分钟后才跌入江中。

江西岸一饭馆的老板娘说，听到巨响，她忙