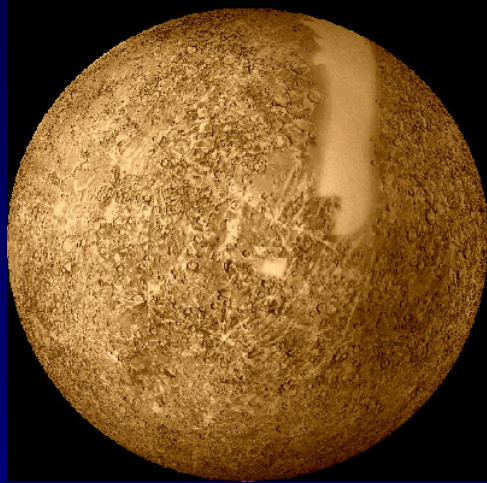




Mercury

水星



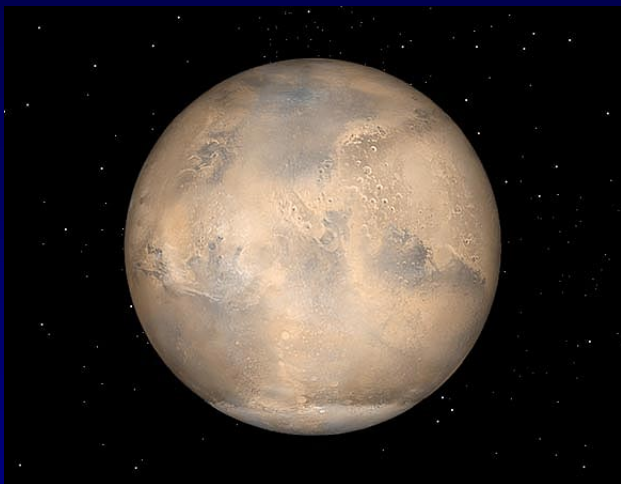
Venus

金星



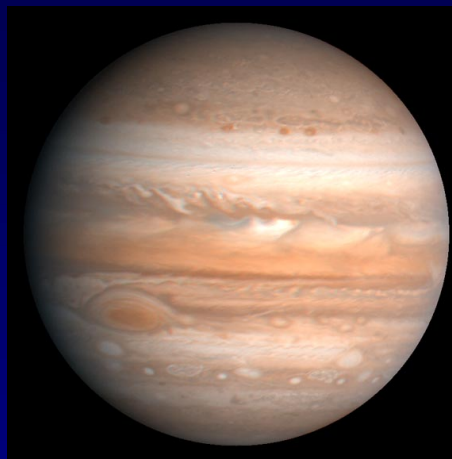
Earth

地球



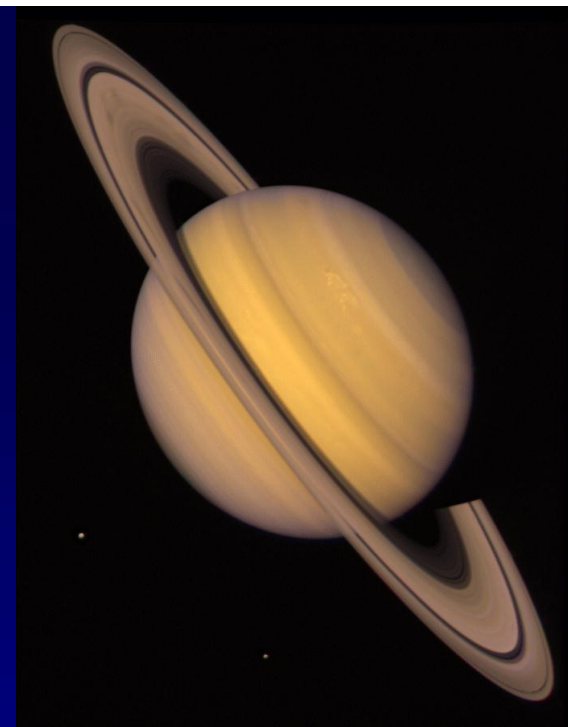
Mars

火星



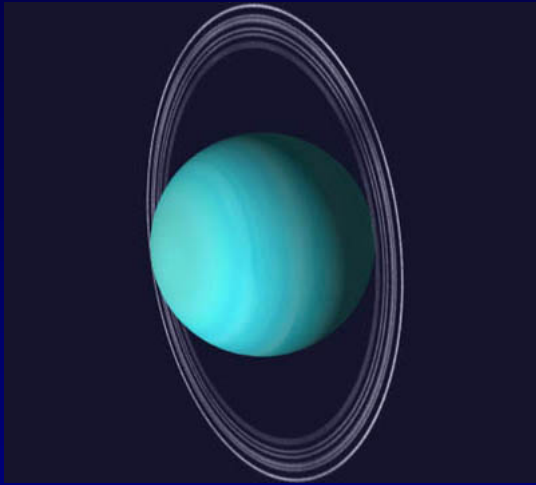
Jupiter

木星



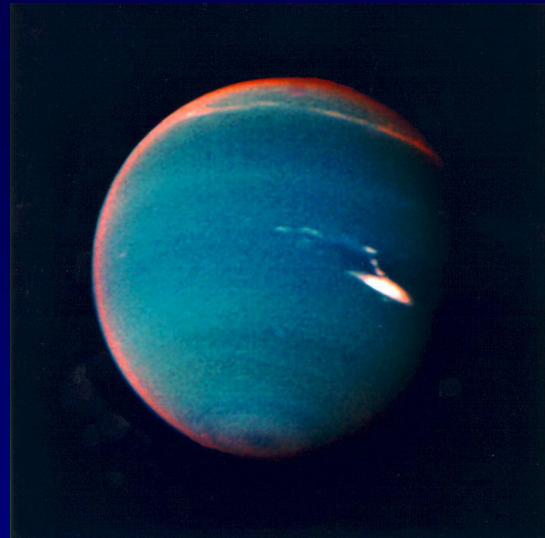
Saturn

土星



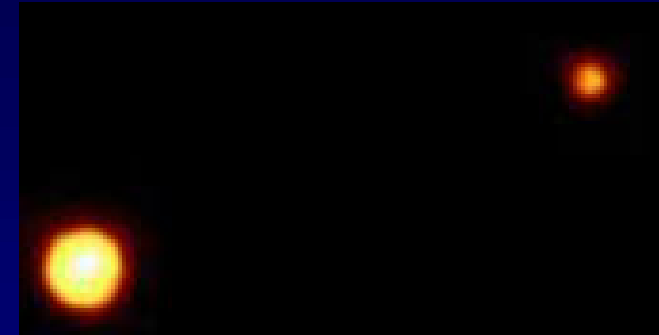
Uranus

天王星



Neptune

海王星



Pluto and its
satellite Charon
(dwarf planet)

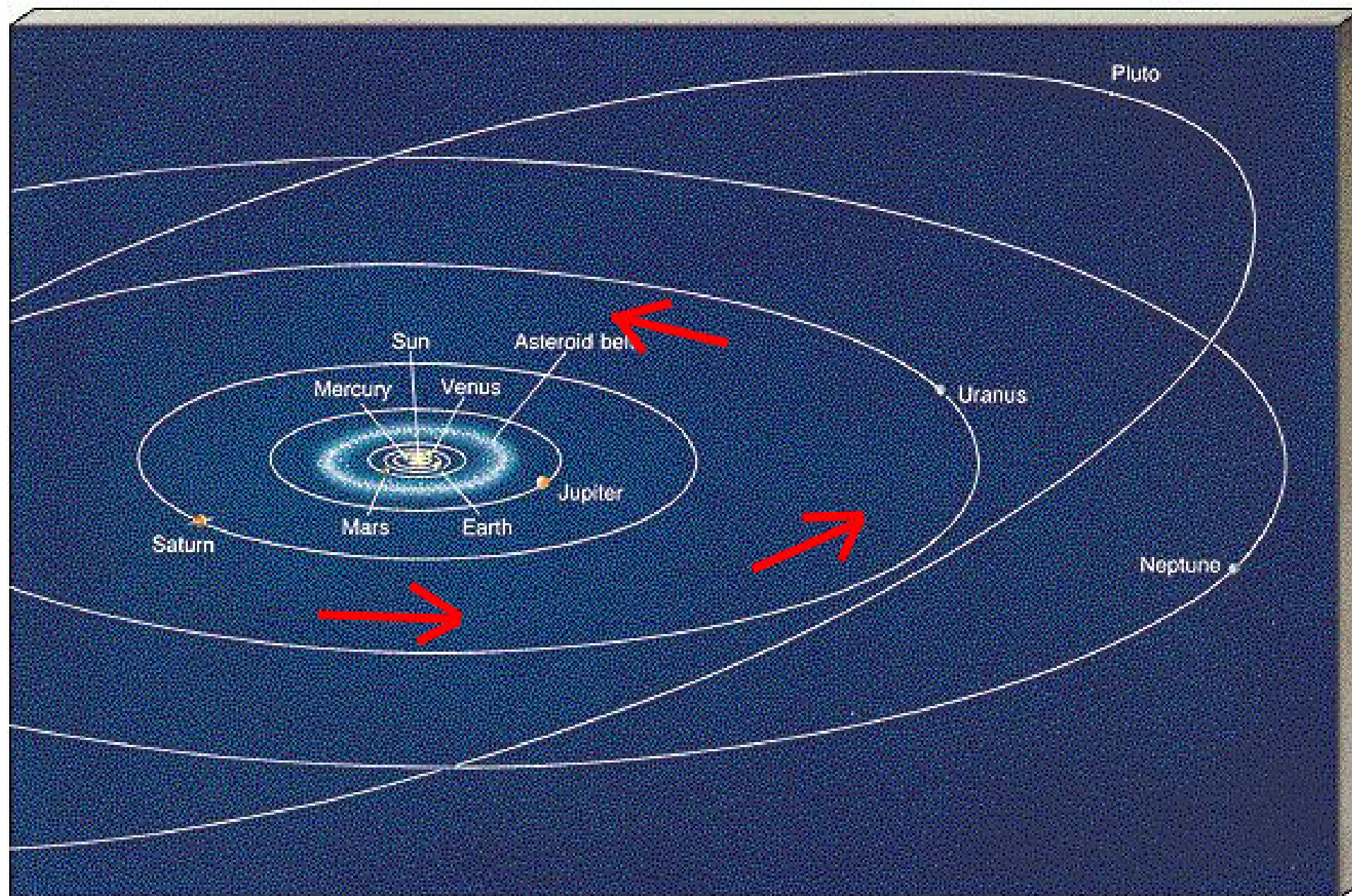
冥王星和冥卫一卡伦

The International Astronomical Union (IAU)
downgraded the status of **Pluto** to that of a "dwarf
planet" on August 24, 2006.

Saturn

土星





Terrestrial planets: Mercury Venus Earth Mars

Gas giants: Jupiter Saturn Uranus Neptune

The **asteroid belt** formed from the **primordial solar nebula** (原始太阳星云) as a group of **planetesimals**(星子), the smaller precursors of the planets, which in turn formed **protoplanets**(原行星) . Between Mars and Jupiter, however, gravitational perturbations from Jupiter imbued the protoplanets with too much orbital energy for them to **accrete**(吸积) into a planet.

Chapter 4

Gravitation

Historical review

The description of planetary motion is a kinematic problem.

Gravitation is closely related to the dynamic problem.

**There were many “world models”
(geocentric, heliocentric, Tycho’s model ...).
All of them need dynamic explanation.**



The frontispiece to
Galileo's *Dialogue
Concerning the Two
World Systems* (1632).

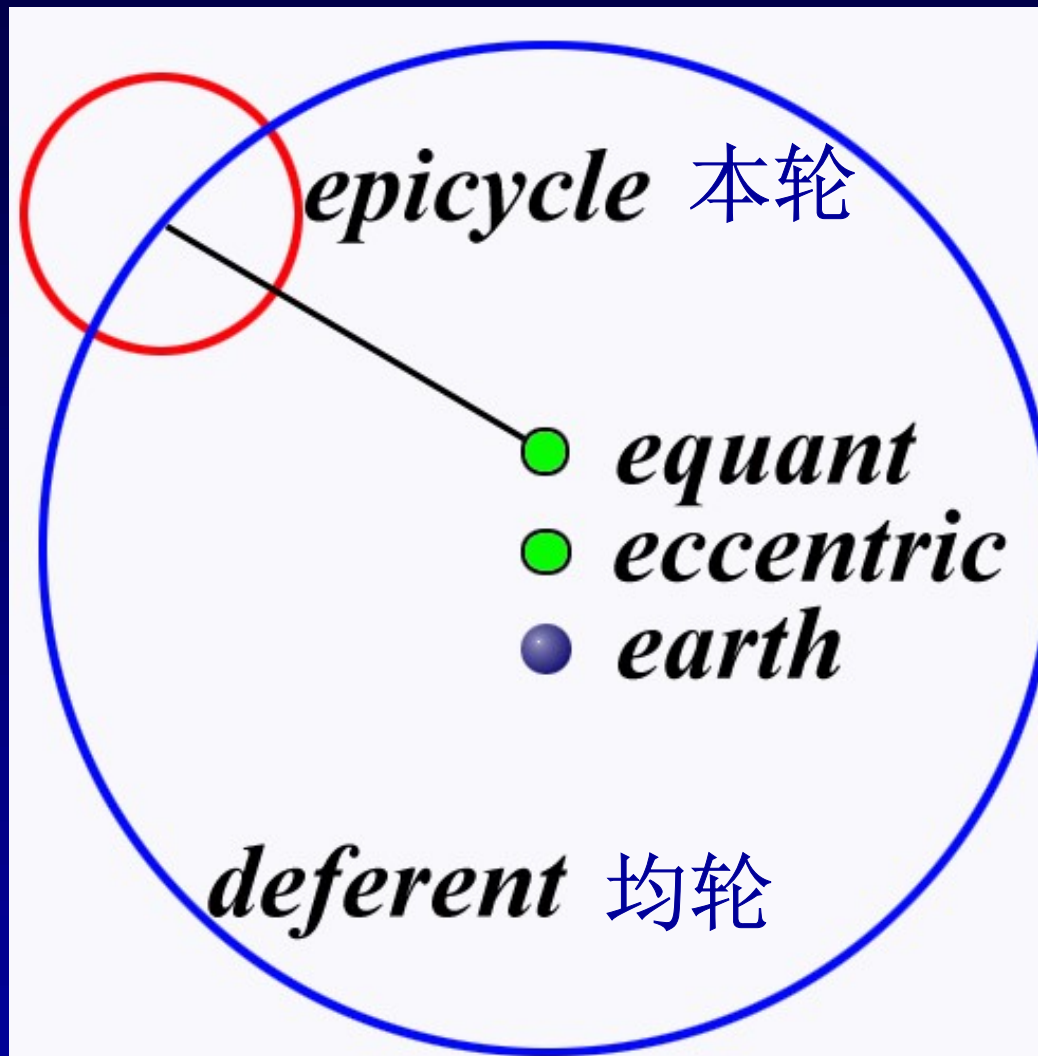
According to the labels,
Copernicus is to the right,
with Aristotle and Ptolemy
at the left; Copernicus was
drawn with Galileo's face,
however



Claudius Ptolemy
(127—152 working in
Alexandria, Egypt)



Nicolaus Copernicus
(1473—1543)



Ptolemy system(70 circles)

均轮

偏心

本轮

Ptolemy system (70 circles)

Copernicus system (46 circles)

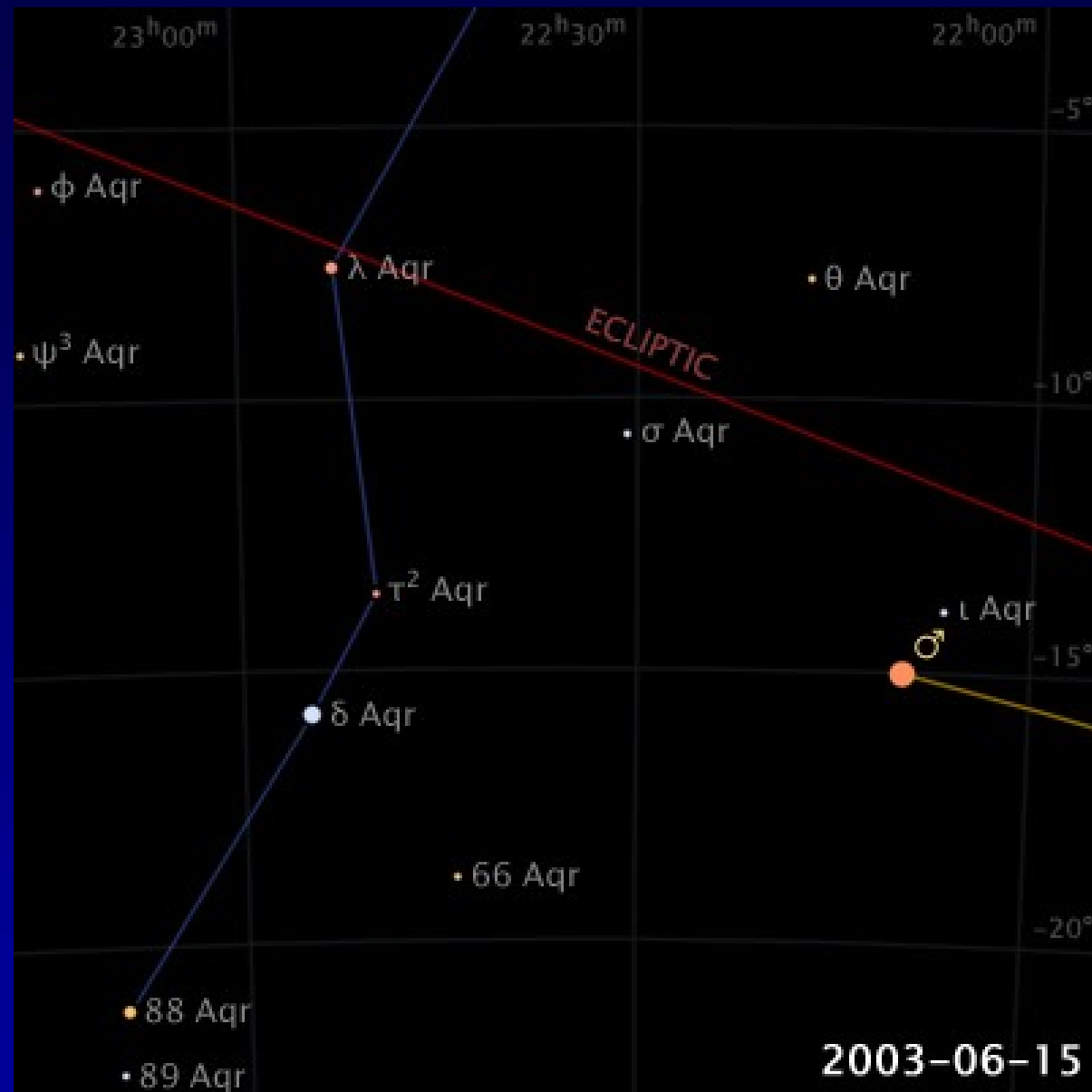
Galileo Galilei (1564—1642)

Tycho Brahe (1546—1601)

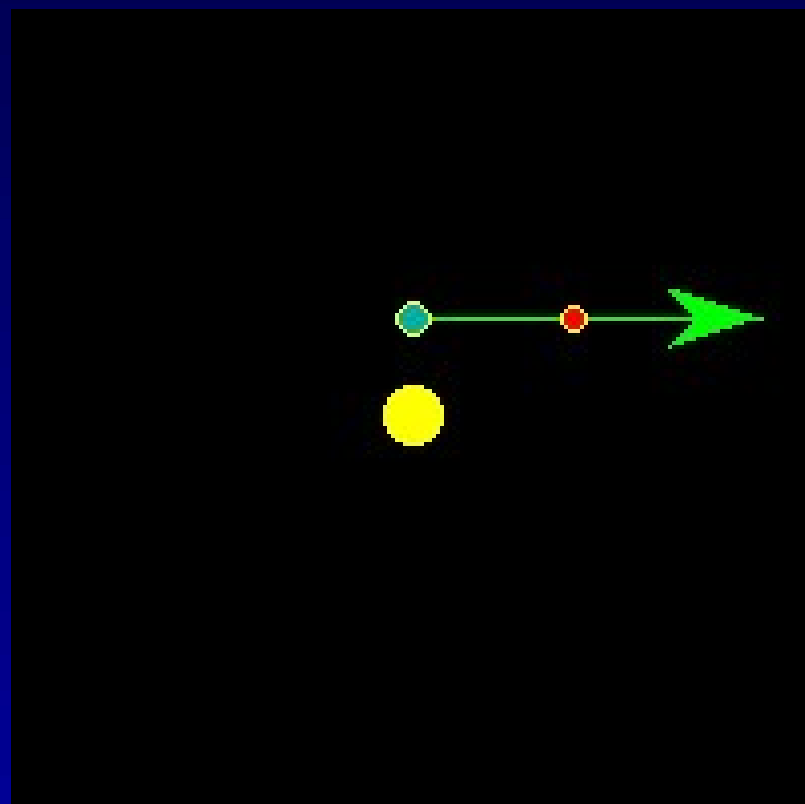
Johannes Kepler (1571—1630)

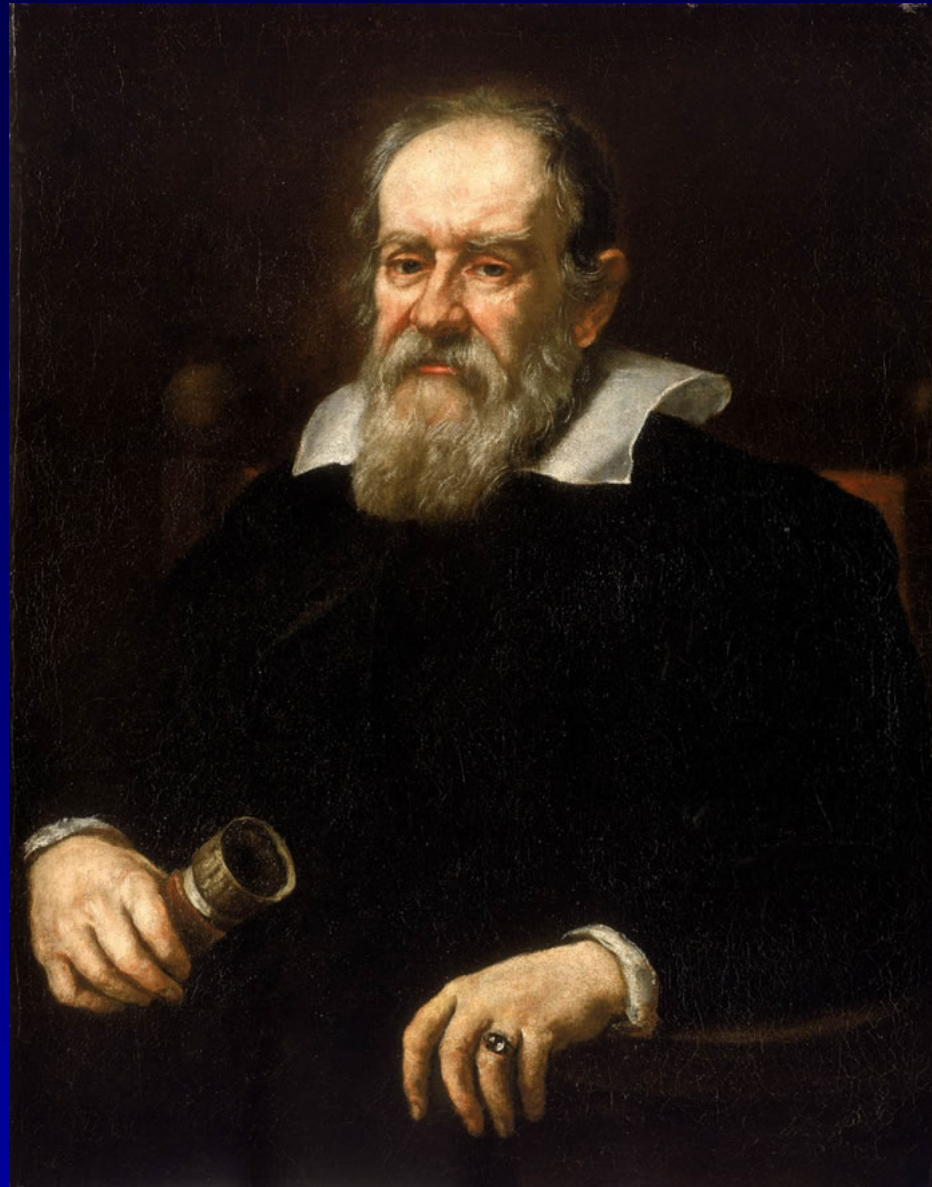
Issac Newton (1642 — 1727)





*Retrograde Motion in the
Copernican System*

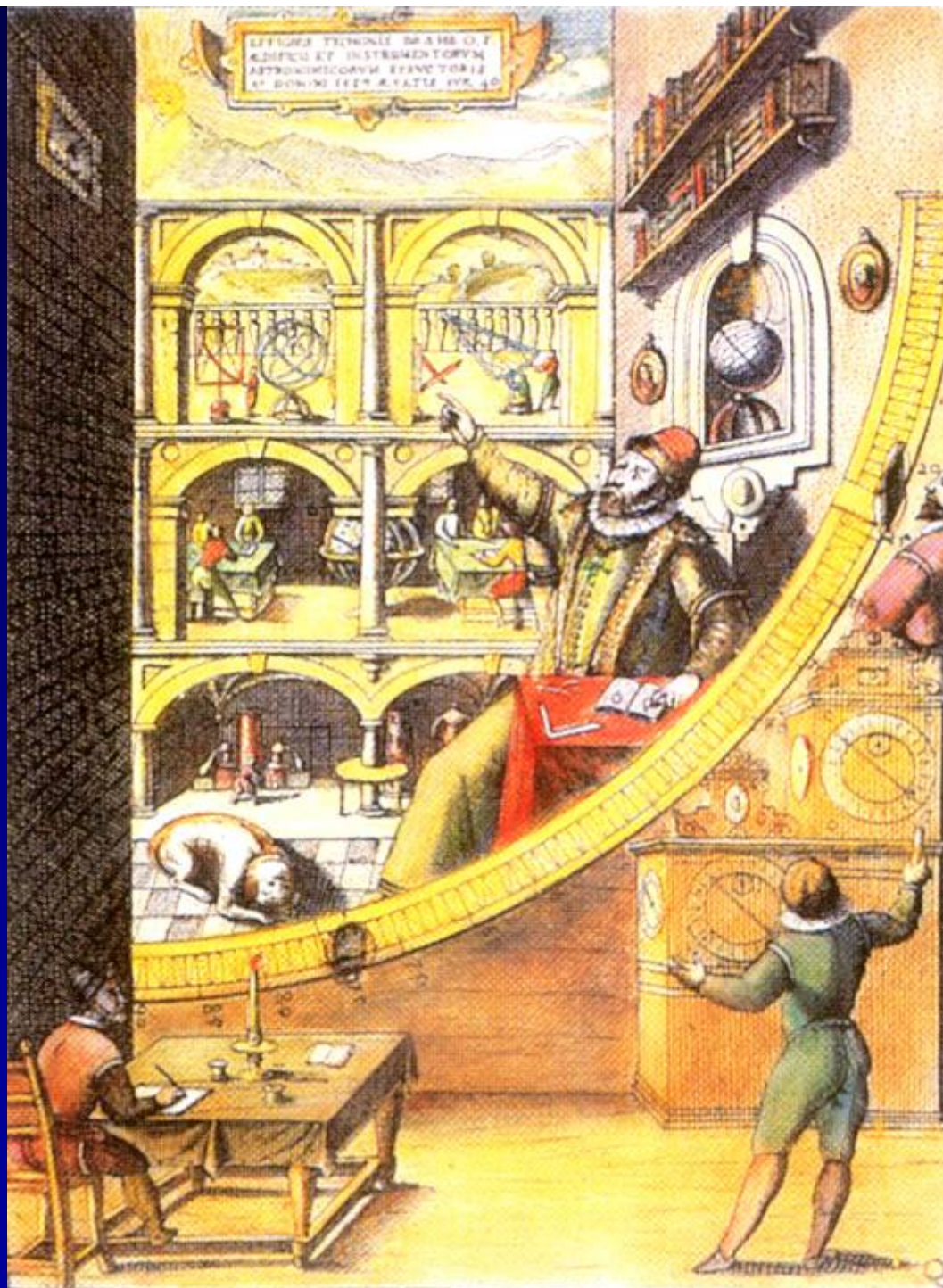




Galileo Galilei (1564—1642)

Tycho's merits to science

- **establishment of scientific experimental procedures**
- **elimination of systematic errors (the flexure of his instruments, refraction)**
- **data with quoted the error: 2 arcmin ($< 0.61\text{mm}/1\text{m}$)**
- **systematically 21 years observation.**

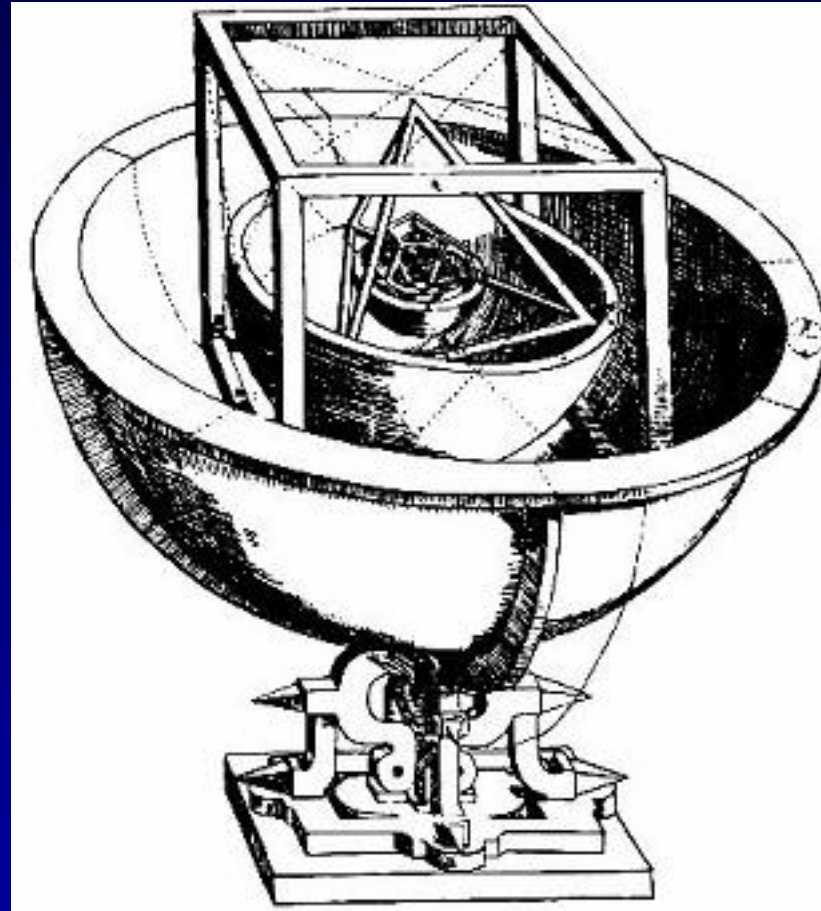


Kepler's nested set:

Saturn—Jupiter—
Mars—Earth—
Venus—Mercury

6-4-12-20-8

regular polyhedra
(polyhedrons)



Saturn	Jupiter	Mars	Earth	Venus	Mercury
cube		dodecahedron		octahedron	
	tetrahedron		icosahedron		

“Pythagorean” or “Platonic” solids

- **Mysterium Cosmographorum (Cosmic Mystery) (1596)**
- **Harmony of the World (1619).**

Kepler's laws

(1) The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

(the law of orbit);

(2) The line joining any planet and the Sun sweeps out equal areas in equal times.

(the law of areas);

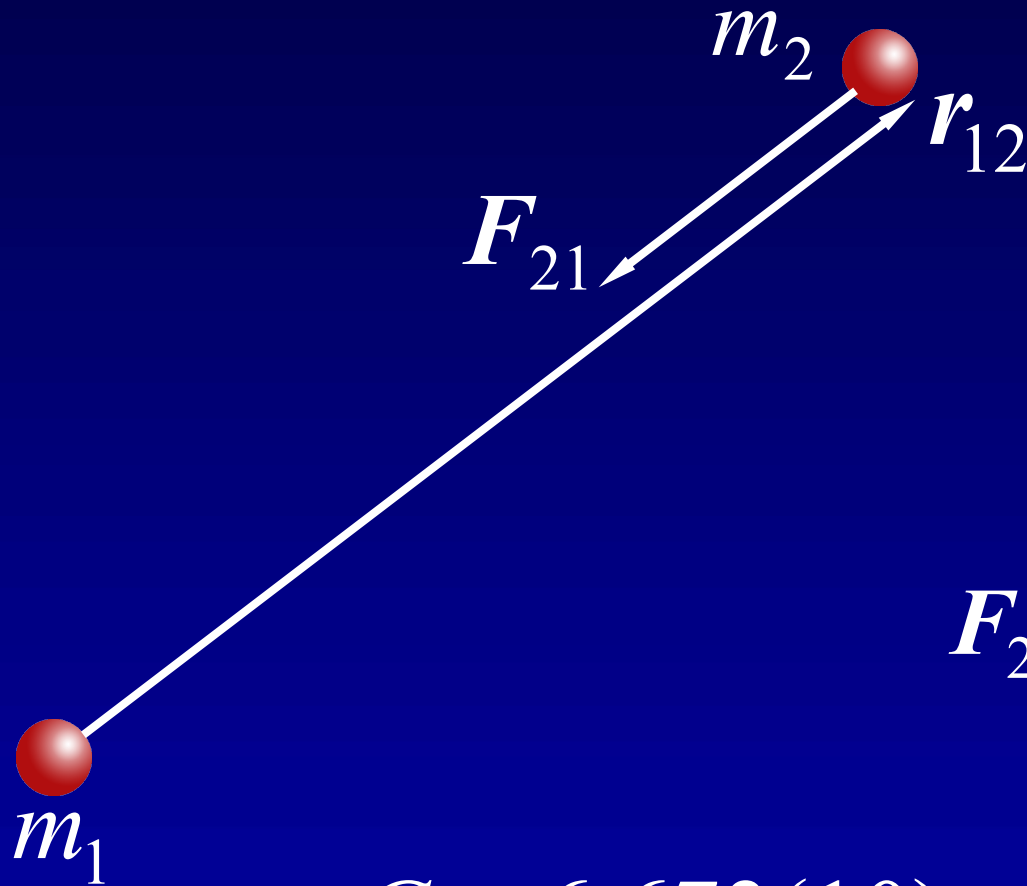
(3) The square of the period of revolution of a planet about the Sun is proportional to the cube of the planet's mean distance of the Sun.

(the law of period or Harmonic law)

In this chapter we will discuss

- **planet motion**
 - **Kepler problem and scattering,**
- **Newton's law of gravitation,**
- **redshift of light in gravitational field,**
- **accretion,**
- **tide and tidal disruption,**
- **...**

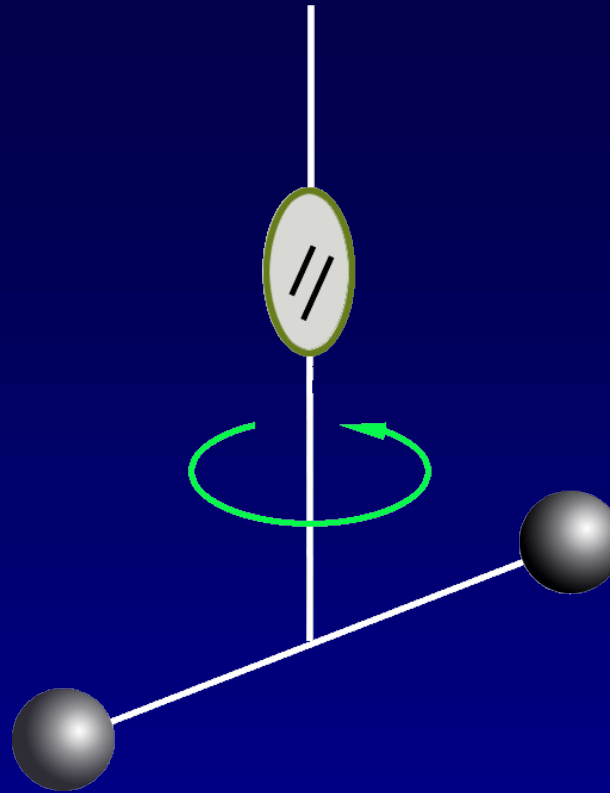
4.1 The law of gravitation



$$\mathbf{F}_{21} = -G \frac{m_1 m_2}{r_{12}^3} \mathbf{r}_{12}$$

$$G = 6.673(10) \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2$$

torsion balance



Henry Cavendish 1798 ‘Weighing the earth’

$$mg = G \frac{mM_E}{R_E^2} \quad M_E = \frac{R_E^2}{G}.$$

Science 5 January 2007:

Vol. 315. no. 5808, pp. 74 - 77

Reports

**Atom Interferometer Measurement of the
Newtonian Constant of Gravity**

J. B. Fixler,¹ G. T. Foster,² J. M. McGuirk,³ M.
A. Kasevich^{1*}

We measured the Newtonian constant of gravity,
 G , using a gravity gradiometer based on atom
interferometer.

$$G = 6.693 \times 10^{-11} \pm 0.027 \times 10^{-11}$$

Department of Physics, Stanford University

In Newton's notebook:

$$F \sim v^2 / r = r\omega^2$$

$$F \sim r\omega^2 \sim r/T^2$$

$$T \sim r^{3/2}, \quad F \sim 1/r^2$$



Kepler's third law

Successes of Newton's law

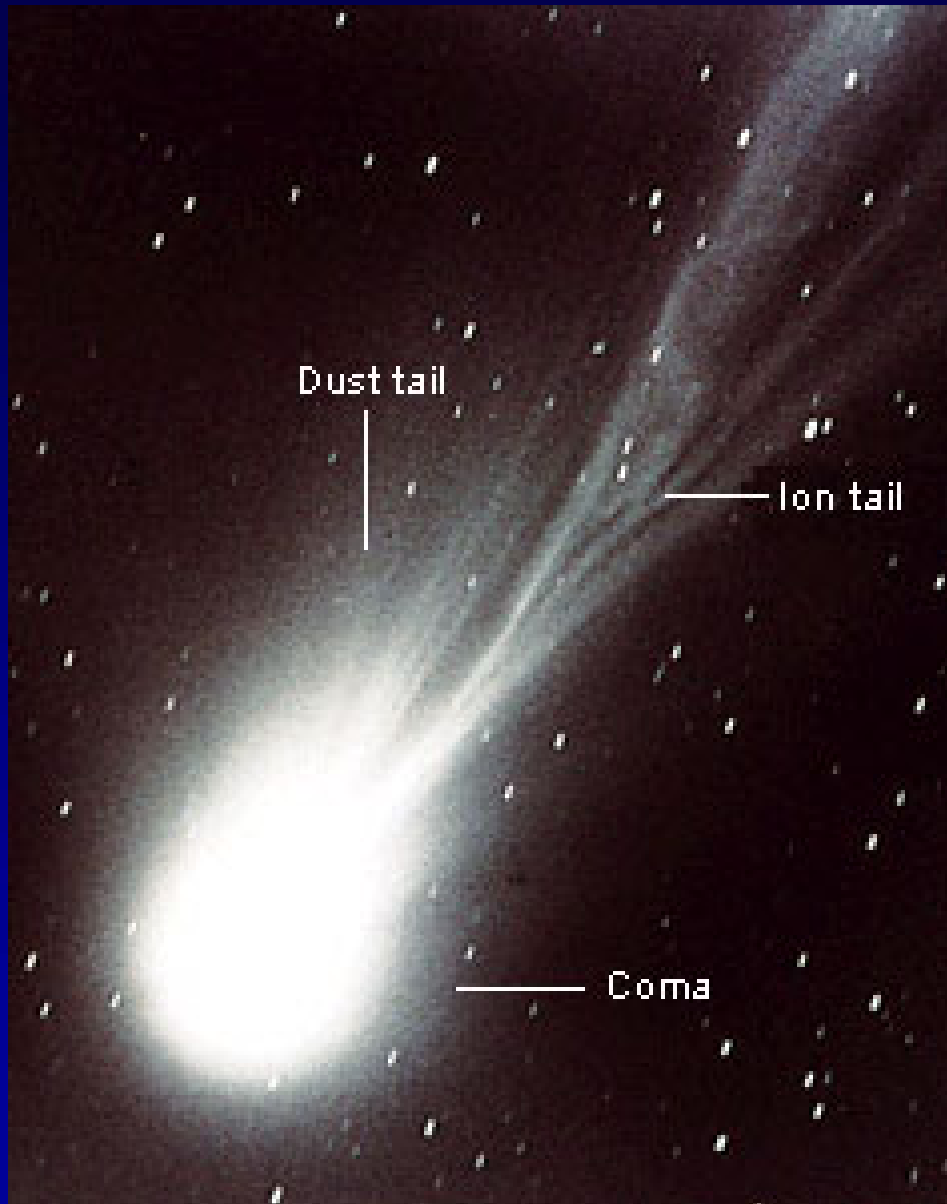
Halley's Comet appeared 1456, 1531, 1607, 1682 with a period about 76 years.

Edmund Halley predicted 1758.

The last time Halley's comet visited Earth, in 1986. Comet Halley isn't officially scheduled to visit Earth again until 2061 when it swings through the inner solar system on its 76-year orbit.



Photographed from Australia on March 13, 1986 (Akira Fujii)



A comet consists of a solid nucleus (core) surrounded by a cloudy atmosphere called the coma(彗发) and one or two tails.

The solar wind from the sun converts some of the comet's gases into ions and these ions stream away from the coma, forming an ion tail .



**The nucleus of Halley's Comet and some of its dust jets.
(© 1987 Max-Planck-Institut für Aeronomie, H.U.Keller)**

海王星

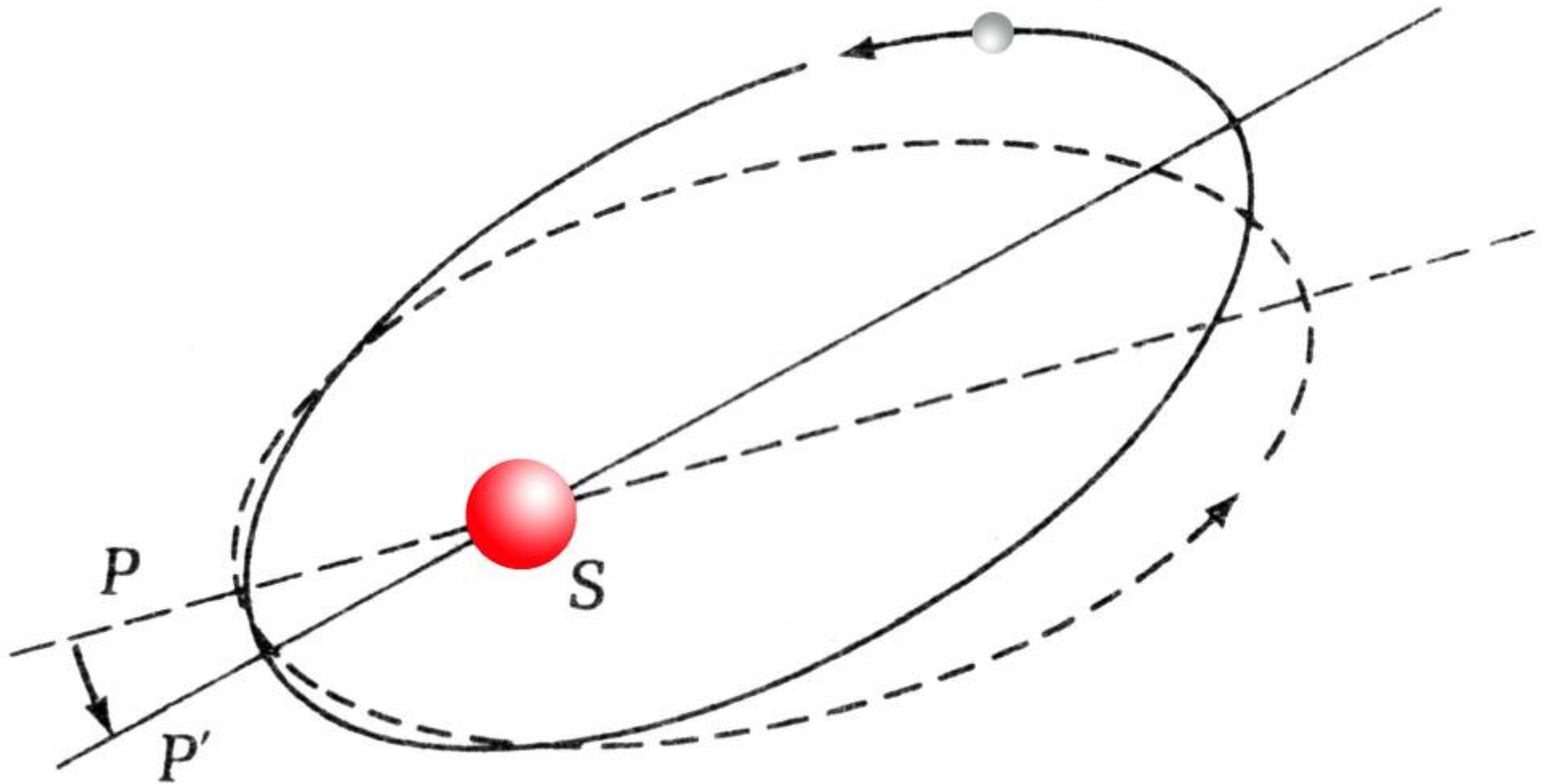
Neptune

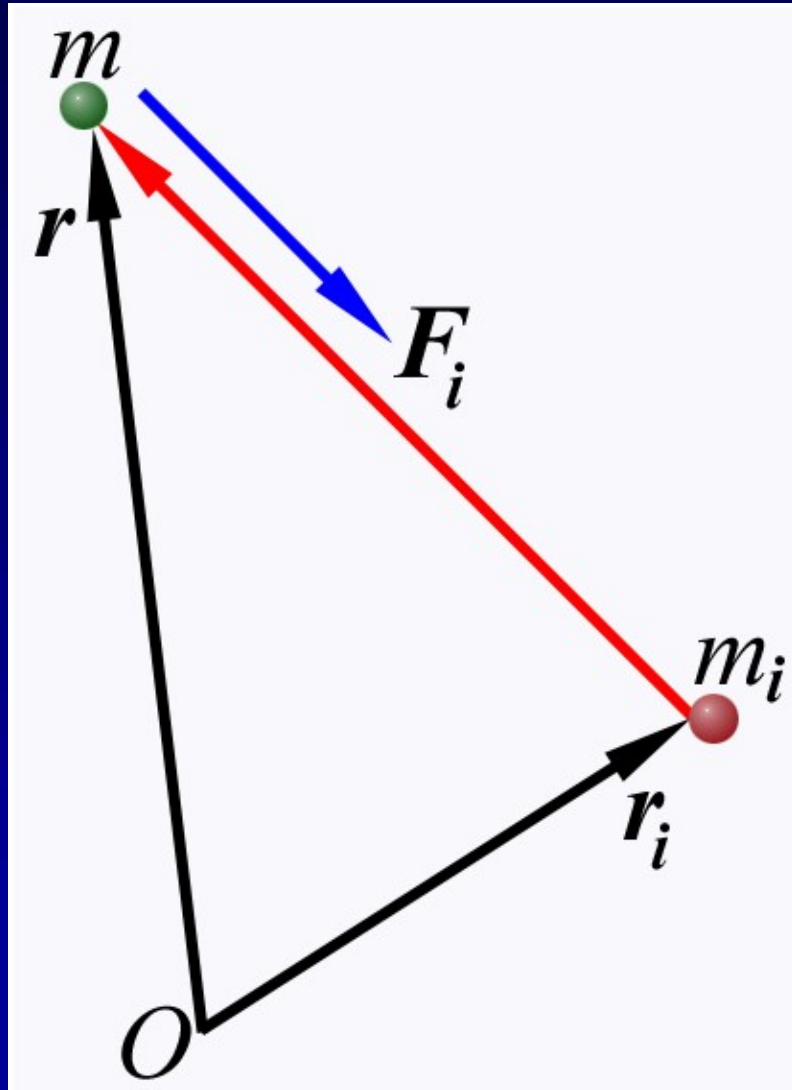
John Adams (1845.10), ignored by Challis and Airy, the leading observational astronomers.

Le Verrier's work (1846.8) was taken seriously by Calle of the Berlin Observatory. On 9. 23

precession of Mercury's perihelion

进动





$$\mathbf{F} = \sum_i \mathbf{F}_i$$

$$= -G \sum_i \frac{m_i m}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i)$$

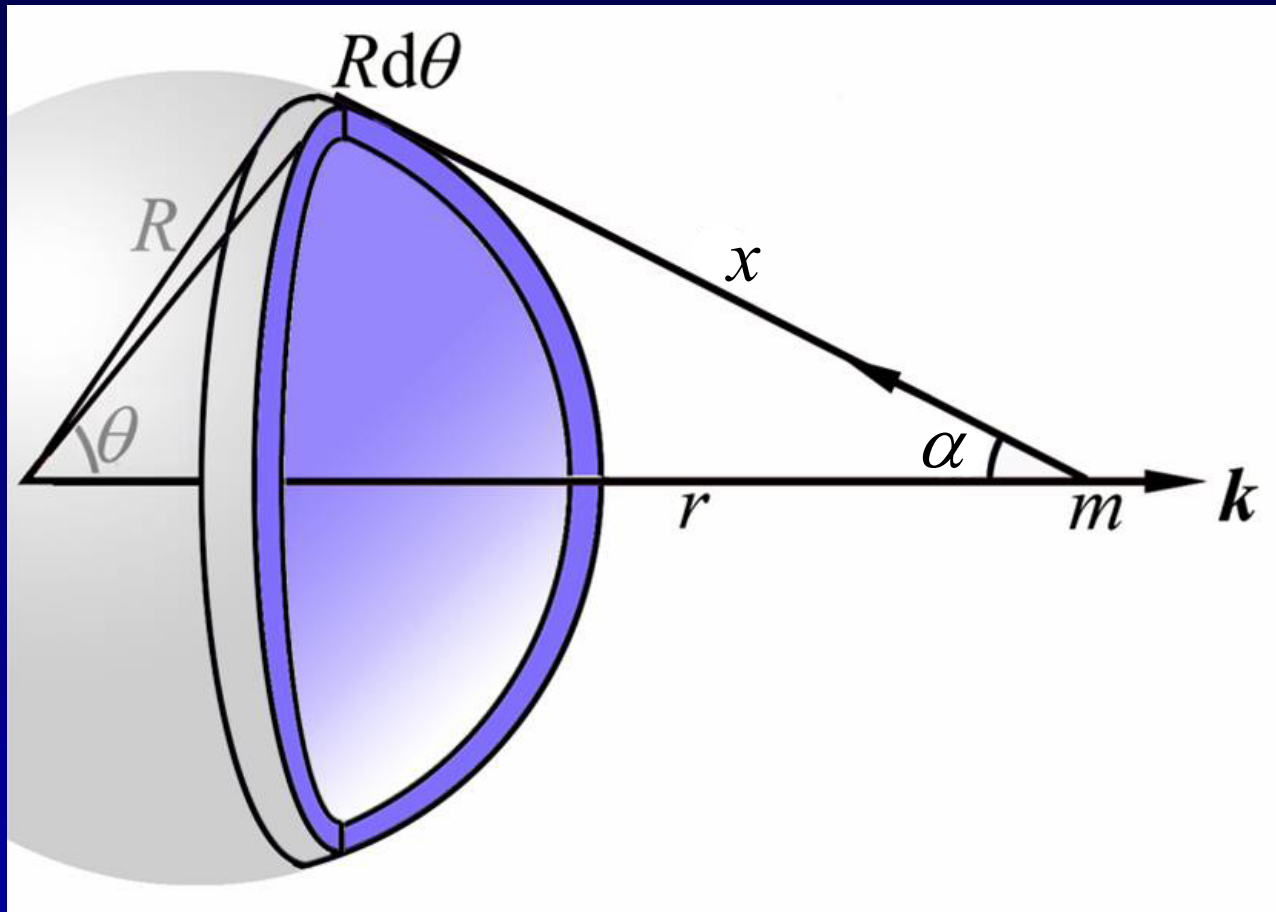
$$= -G \sum_i \frac{(\rho_i \Delta V_i) m}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i)$$

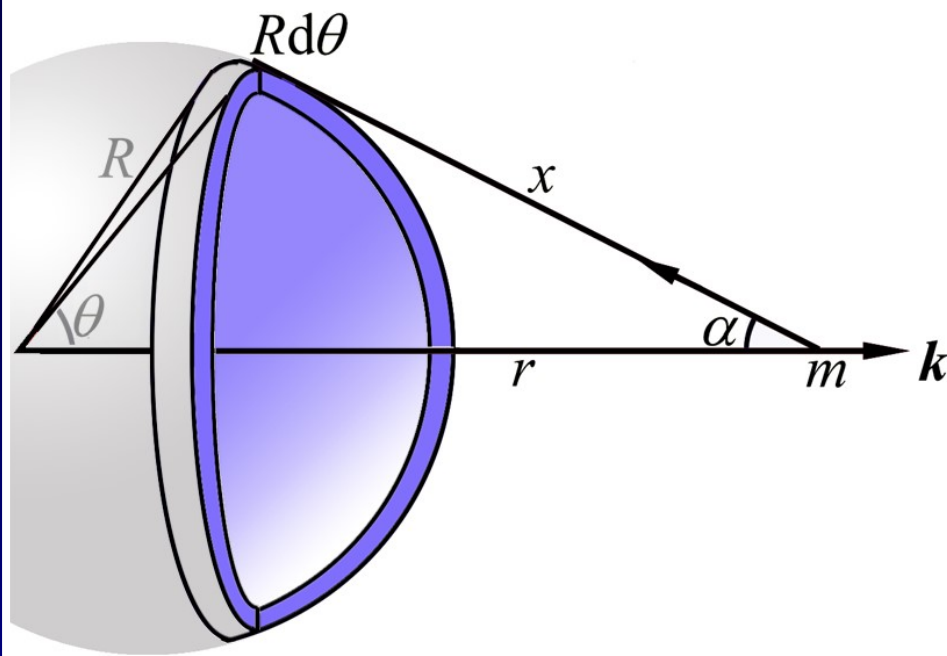
summation index
dummy index



$$= -G \int \frac{(\rho dV') m}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}')$$

Shell~particle





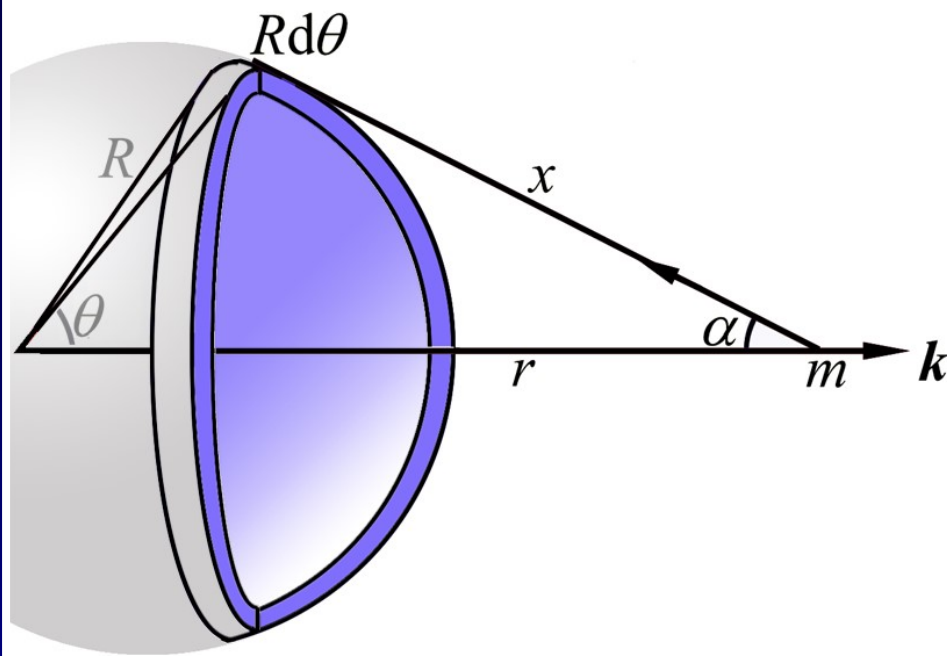
$$\rho = \frac{m'}{4\pi R^2 t}$$

$$dV = 2\pi R \sin \theta t R d\theta$$

dV

$$F = -Fk \quad F = G \int_0^\pi \frac{\overbrace{m \rho 2\pi R^2 t \sin \theta d\theta}^{dV}}{x^2} \cdot \cos \alpha$$

$$= \frac{1}{2} G m' m \int_0^\pi \frac{\sin \theta \cos \alpha d\theta}{x^2}$$



$$x^2 = R^2 + r^2 - 2Rr \cos \theta$$

$$2x dx = 2Rr \sin \theta d\theta$$

$$x \cos \alpha = r - R \cos \theta$$

$$\int_0^\pi \frac{\sin \theta \cos \alpha d\theta}{x^2}$$

$$\int_{r-R}^{r+R} dx$$

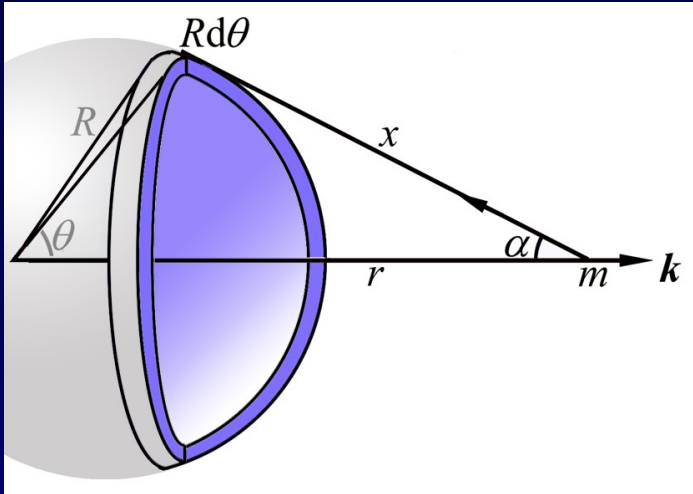
$$x^2 = R^2 + r^2 - 2Rr \cos \theta$$

$$2x dx = 2Rr \sin \theta d\theta$$

$$x \cos \alpha = r - R \cos \theta$$

$$\mathbf{F} = \frac{1}{2} G m' m \int_{|r-R|}^{r+R} \frac{1}{x^2 R r} \left[r - \frac{r^2 + R^2 - x^2}{2r} \right] dx (-\mathbf{k})$$

$$= \begin{cases} -G \frac{m' m}{r^2} \mathbf{k}, & r > R \\ \mathbf{0}, & r < R \end{cases} \quad \begin{array}{l} \int \frac{dx}{x^2} \rightarrow -\frac{1}{x} \\ \int x^0 dx \rightarrow \end{array}$$



$$F = \frac{1}{2} Gm'm \int_0^\pi \frac{\sin \theta \cos \alpha d\theta}{x^2}$$

$$x^2 = R^2 + r^2 - 2Rr \cos \theta$$

$$2x dx = 2Rr \sin \theta d\theta$$

$$x \cos \alpha = r - R \cos \theta$$

$$\sin \theta d\theta = \frac{x dx}{Rr}, \quad \cos \theta = \frac{R^2 + r^2 - x^2}{2Rr}$$

$$x \cos \alpha = r - R \cos \theta = r - \frac{R^2 + r^2 - x^2}{2r}$$

$$F = \frac{1}{2} \frac{Gm'm}{Rr} \int_{|r-R|}^{r+R} \frac{\cos \alpha x dx}{x^2} = \frac{1}{2} \frac{Gm'm}{Rr} \int_{|r-R|}^{r+R} \left(r - \frac{R^2 + r^2 - x^2}{2r} \right) \frac{dx}{x^2}$$

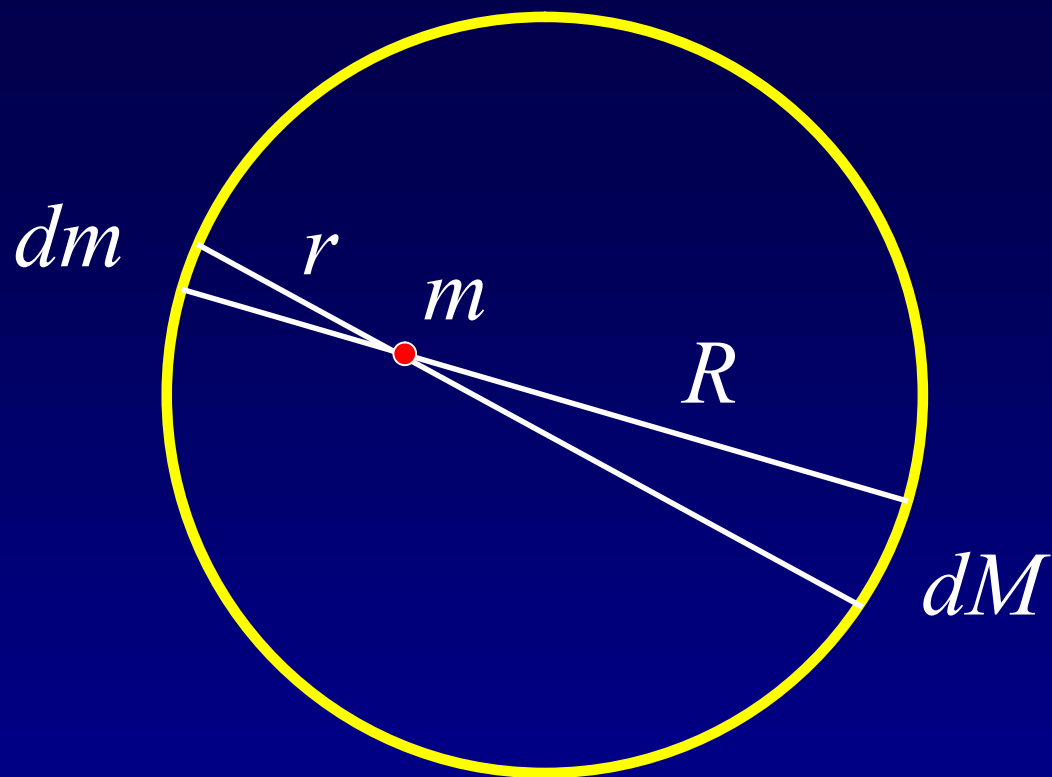
$$\begin{aligned}
F &= \frac{1}{2} \frac{Gm'm}{Rr} \int_{|r-R|}^{r+R} \left(r - \frac{R^2 + r^2 - x^2}{2r} \right) \frac{dx}{x^2} = \frac{1}{2} \frac{Gm'm}{Rr} \int_{|r-R|}^{r+R} \left(\frac{r^2 - R^2 + x^2}{2r} \right) \frac{dx}{x^2} \\
&= \frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \int_{|r-R|}^{r+R} \frac{dx}{x^2} + \frac{1}{4} \frac{Gm'm}{Rr^2} \int_{|r-R|}^{r+R} dx \\
&= -\frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \left(\frac{1}{r+R} - \frac{1}{|r-R|} \right) + \frac{1}{4} \frac{Gm'm}{Rr^2} (r+R - |r-R|).
\end{aligned}$$

For $r > R$, we have

$$\begin{aligned}
F &= -\frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \left(\frac{1}{r+R} - \frac{1}{r-R} \right) + \frac{1}{4} \frac{Gm'm}{Rr^2} (r+R - r+R) \\
&= -\frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \left[\frac{r-R - (r+R)}{r^2 - R^2} \right] + \frac{1}{2} \frac{Gm'm}{r^2} = \frac{Gm'm}{r^2}.
\end{aligned}$$

For $r < R$,

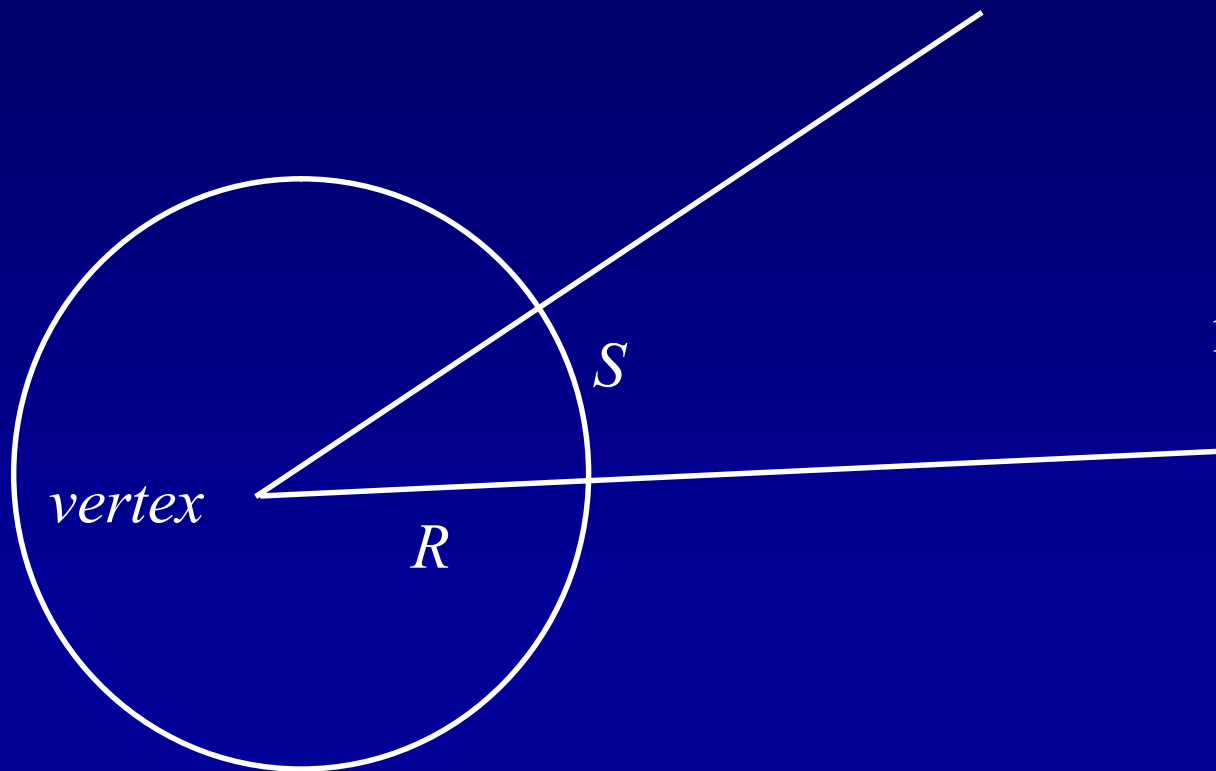
$$\begin{aligned} F &= -\frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \left(\frac{1}{r+R} - \frac{1}{R-r} \right) + \frac{1}{4} \frac{Gm'm}{Rr^2} (r+R - R+r) \\ &= -\frac{1}{4} Gm'm \frac{r^2 - R^2}{Rr^2} \left[\frac{r-R+(r+R)}{r^2 - R^2} \right] + \frac{1}{2} \frac{Gm'm}{Rr} = 0. \end{aligned}$$



$$dm \propto r^2$$

$$dM \propto R^2$$

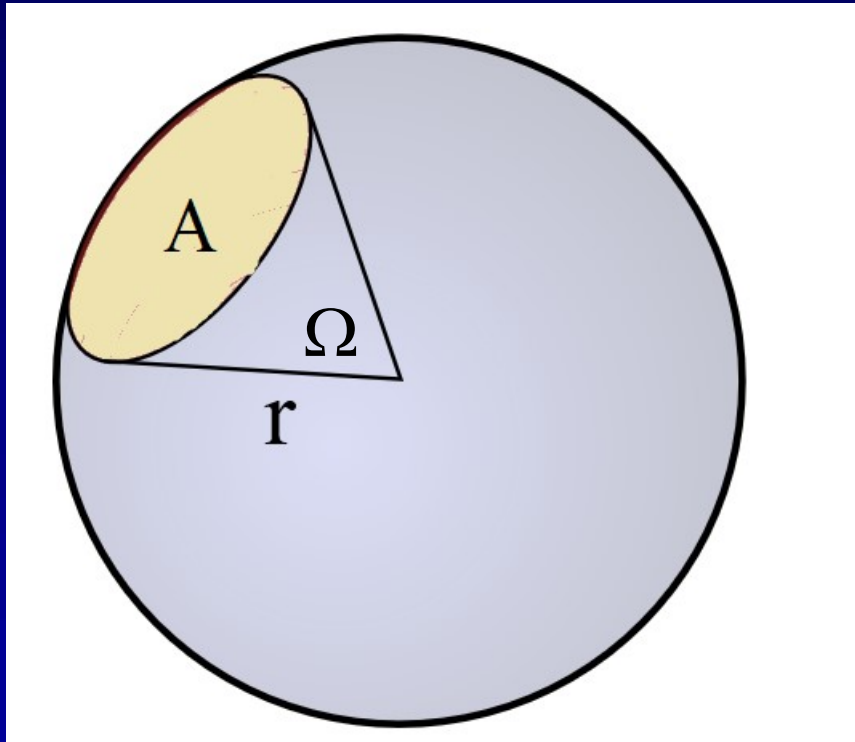
A planar angle is the space between two rays on a plane that have a common endpoint.



$$\alpha = \frac{S}{R}$$

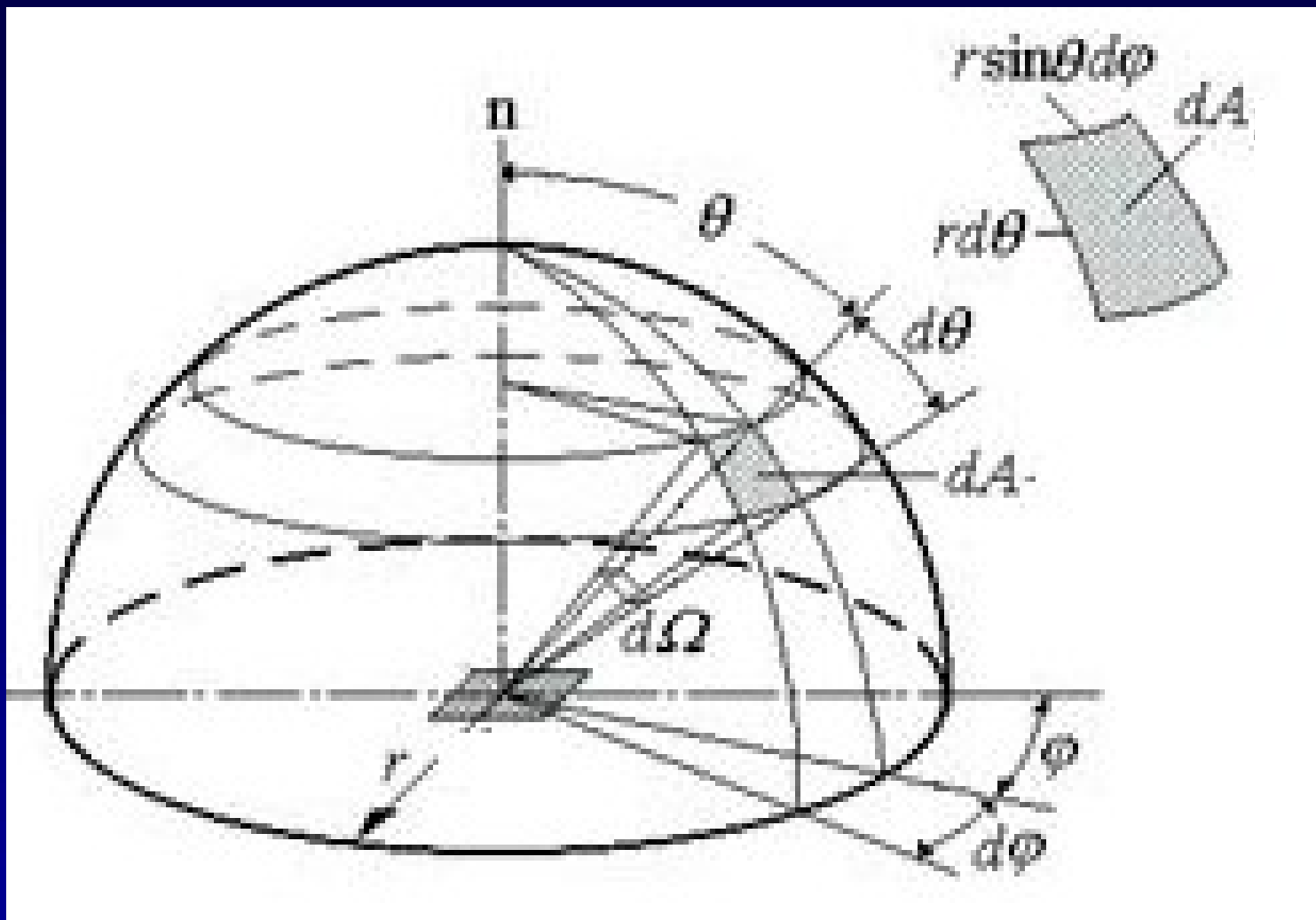
radian (弧度)

A solid angle is the space enclosed by a conical surface.



$$\Omega = \frac{A}{r^2}$$

Steradian (立体弧度)



$$dA = rd\theta \cdot \underline{r \sin \theta} d\phi, \quad d\Omega = \frac{dA}{r^2} = \sin \theta d\theta d\phi.$$

distance to the z-axis

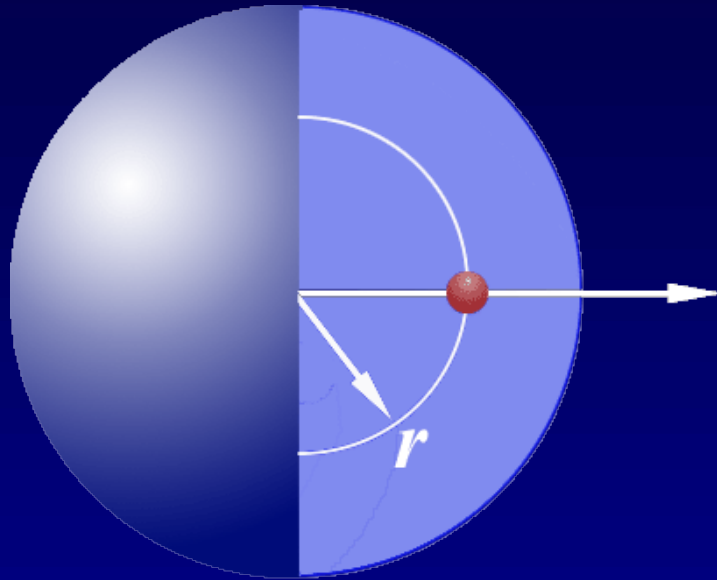
$$\mathbf{F} = \begin{cases} -G \frac{m' m}{r^2} \mathbf{k}, & r > R \\ \mathbf{0}, & r < R \end{cases}$$

Outside a **sphere**, say the earth,

$$\mathbf{F} = -G \frac{m_{\oplus} m}{r^2} \mathbf{r} = m \mathbf{g}$$

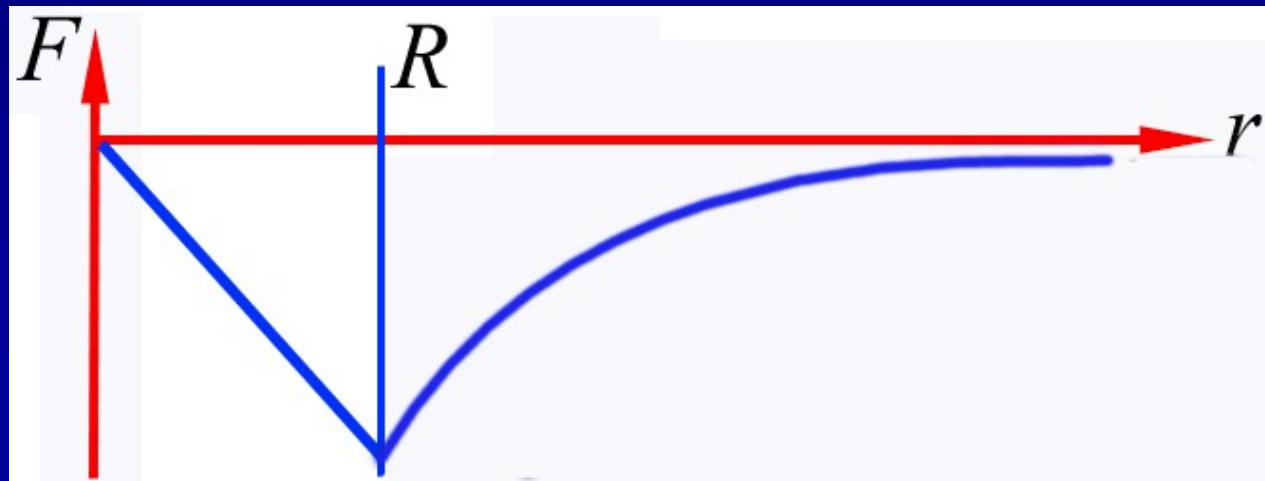
$$G \frac{m_{\oplus}}{r^2} = g \qquad \frac{dg}{g} = -2 \frac{dr}{r}$$

Inside a sphere

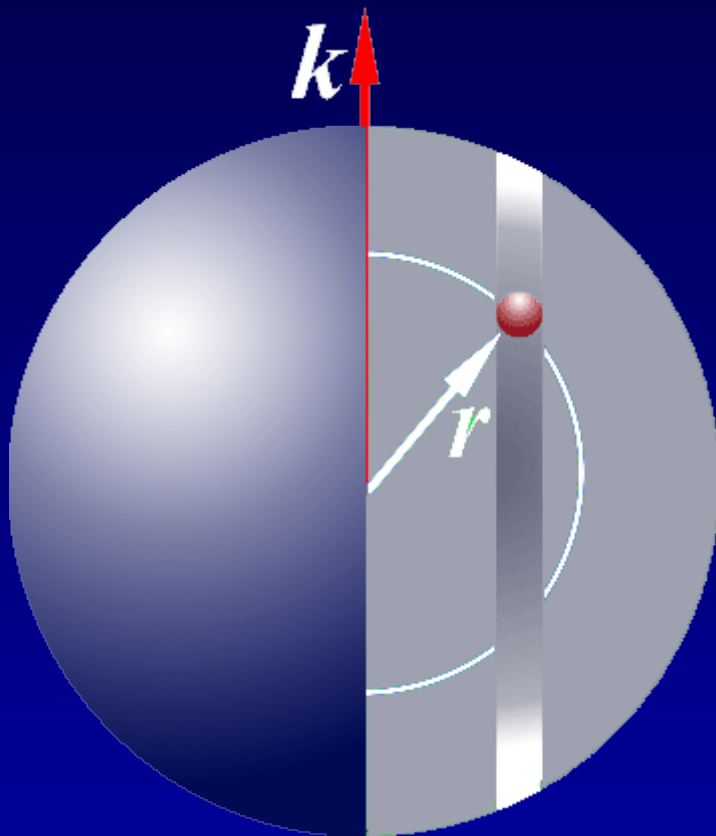


$$F = -G \frac{\left(m' \frac{r^3}{R^3} \right) \cdot m}{r^2}$$

$$= -G \frac{m' m}{R^2} \frac{r}{R}$$



Example 4.1 a chute in sphere



$$F_r = -\frac{Gm'm}{R^3}r$$

$$F_z = -\frac{Gm'm}{R^3}z$$

$$m\ddot{z} = -\frac{Gm'm}{R^3}z$$

Assignment:4.4

Mercury is named after the Roman deity Mercury, the messenger of the gods.

Venus is named after the Roman goddess of love and beauty.

Earth is the planet we live on.

Mars is named the Roman god of war.

Jupiter is named after the god of the sky and thunder and king of the gods in Ancient Roman religion and mythology.

Saturn is named after the Roman god of agriculture.

Uranus is the only planet whose name is derived directly from a figure from Greek mythology, from the Latinised version of the Greek god of the sky Ouranos.

Neptune is named after the Roman god of the sea.

Pluto (dwarf planet) is named after the Roman god of the underworld and its satellite Charon is named after the mythological boatman who ferried souls across the river.

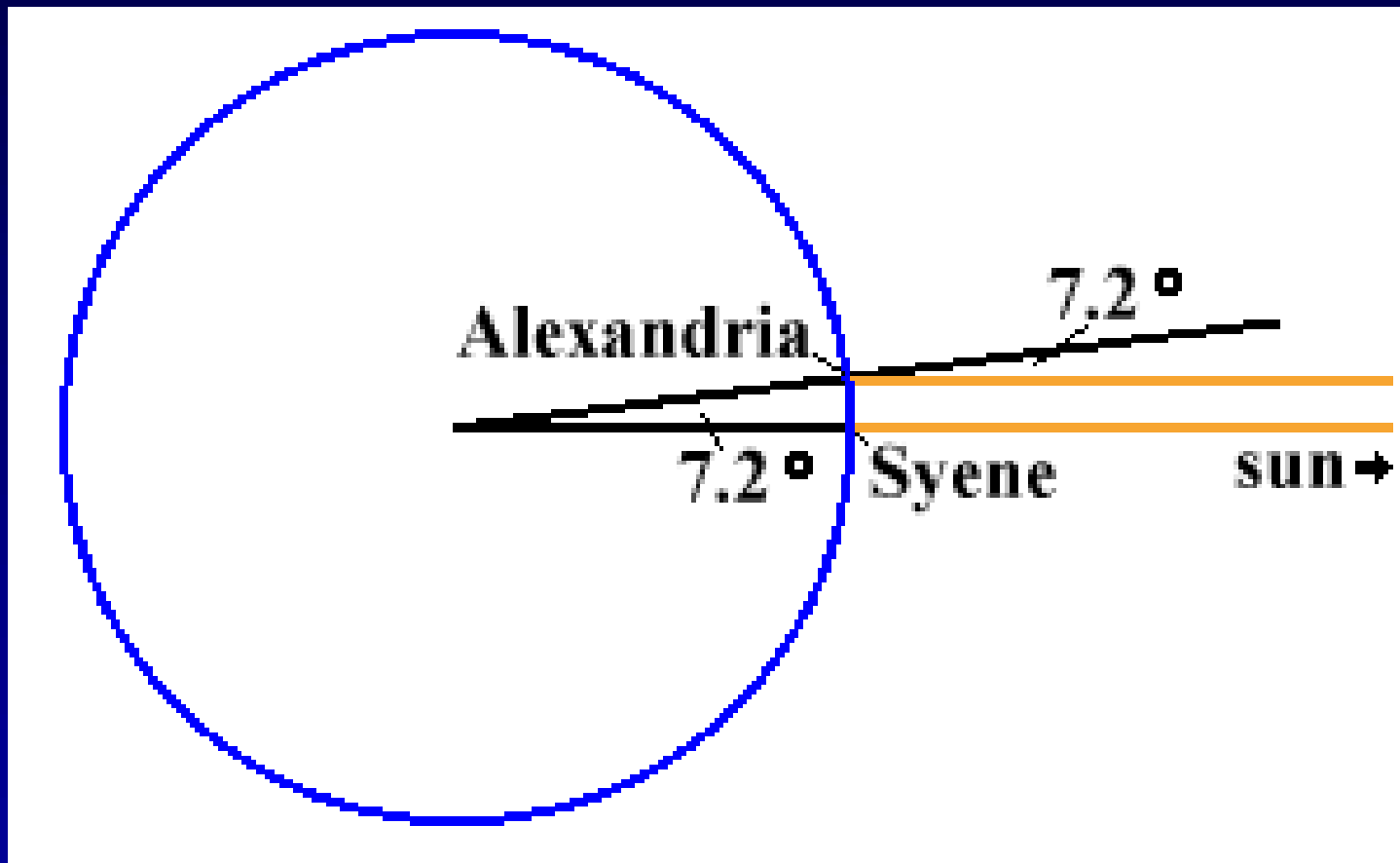
Aristotle (4th century BC) showed that the Earth is spherical.

Eratosthenes (ca. 200 BC) determined the size of the Earth.

No one laughed at Christopher Columbus when he said the world was round.

lunar eclipse





Eratosthenes calculated the circumference of the Earth without leaving Egypt. Eratosthenes knew that on the summer solstice at local noon in the Ancient Egyptian city of Swenet (known in Greek as Syene, and in the modern day as Aswan) on the Tropic of Cancer, the sun would appear at the zenith, directly overhead (he had been told that the shadow of someone looking down a deep well would block the reflection of the Sun at noon). He also knew, from measurement, that in his hometown of Alexandria, the angle of elevation of the sun was $1/50$ th of a circle ($7^{\circ} 12'$) south of the zenith on the solstice noon. Assuming that the Earth was spherical (360°), and that Alexandria was due north of Syene, he concluded that the meridian arc distance from Alexandria to Syene must therefore be $1/50 = 7^{\circ} 12' / 360^{\circ}$, and was therefore $1/50$ of the total circumference of the Earth. His knowledge of the size of Egypt after many generations of surveying trips for the Pharaonic bookkeepers gave a distance between the cities of 5000 stadia (about 500 geographical miles or 800 km). This distance was corroborated by inquiring about the time that it takes to travel from Syene to Alexandria by camel. He rounded the result to a final value of 700 stadia per degree, which implies a circumference of 252,000 stadia. The exact size of the stadion he used is frequently argued. The common Attic stadion was about 185 m,[9] which would imply a circumference of 46,620 km, i.e. 16.3% too large. However, if we assume that Eratosthenes used the "Egyptian stadion"[10] of about 157.5 m, his measurement turns out to be 39,690 km, an error of less than 2%.

In 1821, Alexis Bouvard published astronomical tables of the orbit of Neptune's neighbor Uranus.[23] Subsequent observations revealed substantial deviations from the tables, leading Bouvard to hypothesize that an unknown body was perturbing the orbit through gravitational interaction.[24] In 1843, John Couch Adams began work on the orbit of Uranus using the data he had. Via James Challis, he requested from Sir George Airy, the Astronomer Royal, who sent the data in February 1844. Adams continued to work on this in 1845–46 and produced several different estimates of a new planet, but did not respond to a request from Airy about the orbit of Uranus.

In 1845–46, Urbain Le Verrier, independently of Adams, developed his own calculations but also experienced difficulties in stimulating any enthusiasm in his compatriots. In June 1846, upon seeing Le Verrier's first published estimate of the planet's longitude and its similarity to Adams's estimate, Airy persuaded Cambridge Observatory director James Challis to search for the planet. Challis vainly scoured the sky throughout August and September.

Meantime, Le Verrier by letter urged Berlin Observatory astronomer Johann Gottfried Galle to search with the observatory's refractor. Heinrich d'Arrest, a student at the observatory, suggested to Galle that they could compare a recently drawn chart of the sky in the region of Le Verrier's predicted location with the current sky to seek the displacement characteristic of a planet, as opposed to a fixed star. The very evening of the day of receipt of Le Verrier's letter on September 23, 1846, Neptune was discovered within 1° of where Le Verrier had predicted it to be, and about 12° from Adams' prediction. Challis later realized that he had observed the planet twice in August, failing to identify it owing to his casual approach to the work.

4.2 Gravitational potential energy

m —particle m' —particle, shell, sphere

$$W = \int \left(-G \frac{m'm}{r^3} \mathbf{r} \right) \cdot d\mathbf{r} = -\Delta U$$

m outside of sphere

$$= -Gm'm \int_{r_i}^r \frac{1}{r^2} dr$$

$$= - \left[\left(-G \frac{m'm}{r} \right) - \left(-G \frac{m'm}{r_i} \right) \right]$$

$$U = -G \frac{m' m}{r} + U_0$$

● $U(\infty) = 0, \quad U(r) = -G \frac{m_{\oplus} m}{r}$

● $U(R_{\oplus}) = 0, \quad U = -\frac{G m_{\oplus} m}{(R_{\oplus} + z)} + \frac{G m_{\oplus} m}{R_{\oplus}} \xrightarrow{?}$

$$U = \frac{G m_{\oplus} m}{R_{\oplus}} \left[1 - \frac{R_{\oplus}}{R_{\oplus} + z} \right]$$

$$U = mgz$$

$$= \frac{G m_{\oplus} m}{R_{\oplus}} \frac{z}{R_{\oplus} + z} = \frac{G m_{\oplus} m}{R_{\oplus}^2} \frac{z}{1 + \frac{z}{R_{\oplus}}}$$

$$U = -G \frac{m' m}{r} + U_0$$

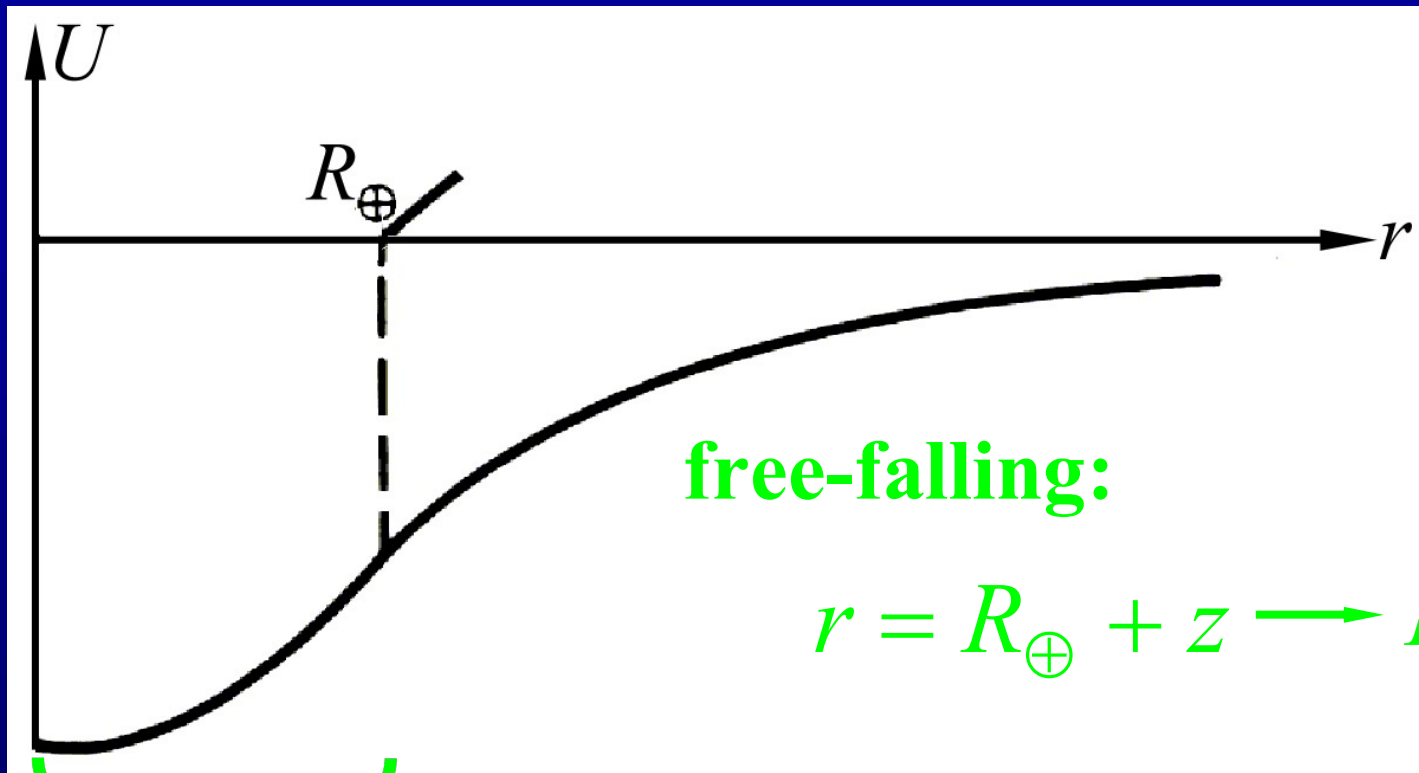
● $U(\infty) = 0, \quad U(r) = -G \frac{m_{\oplus} m}{r}$

● $U(R_{\oplus}) = 0, \quad U = -\frac{G m_{\oplus} m}{(R_{\oplus} + z)} + \frac{G m_{\oplus} m}{R_{\oplus}} \xrightarrow{?}$

$$U = \frac{G m_{\oplus} m}{R_{\oplus}} \left[1 - \frac{R_{\oplus}}{R_{\oplus} + z} \right]$$

$$= mgz \frac{1}{1 + \frac{z}{R_{\oplus}}} \approx mgz \left[1 - z/R_{\oplus} \right]$$

$$U = mgz$$



?

$$-G \frac{\cancel{m_\oplus} \cancel{m}}{R_\oplus + z} + G \frac{\cancel{m_\oplus} \cancel{m}}{R_\oplus} = \frac{1}{2} \cancel{m} v^2$$

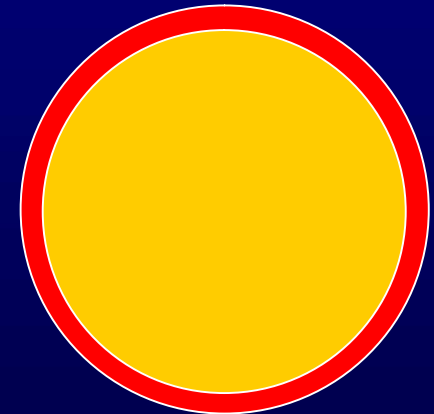
$$v^2 = 2gz \left(1 - \frac{z}{R_\oplus} \right)$$

Self-energy

$$m \frac{r^3}{R^3} (\text{core}), \quad \rho 4\pi r^2 dr = \frac{3r^2 m}{R^3} dr (\text{shell})$$

$$U_s = -G \int_0^R \boxed{\frac{mr^3}{R^3}} \frac{1}{r} \boxed{\frac{3r^2 m}{R^3}} dr = -\frac{3}{5} G \frac{m^2}{R}$$

Core to all particles on the shell

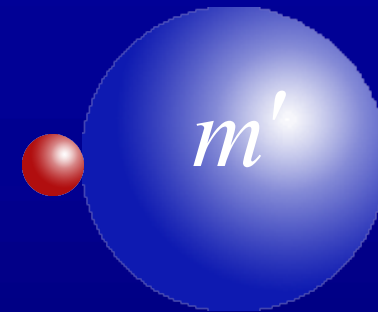


accretion

$$\Delta E_a = G \frac{m' m}{R} \quad \text{released}$$

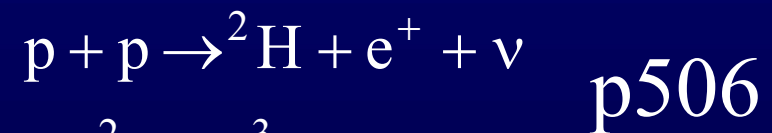
$$\frac{\Delta E}{m} = G \frac{m'}{R} \quad \text{if } m' \approx m_{\odot} \quad R_* \sim 10 \text{ km}$$

$$\frac{\Delta E}{m} \sim \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{10 \times 10^3} \text{ J/kg} \sim 10^{16} \text{ J/kg}$$



In nuclear fusion

$$\frac{\Delta E_f}{m} = 6.3 \times 10^{14} \text{ J/kg}$$



p506

orbiting

$$m \frac{v^2}{\cancel{r}} = G \frac{m' m}{\cancel{r^2}}$$

$$U = -G \frac{m' m}{r}$$

$$E_k = \frac{1}{2} m v^2 = \frac{1}{2} G \frac{m' m}{r} = -\frac{1}{2} U$$

$$E = E_k + U = \frac{1}{2} U = -\frac{G m' m}{2r} < 0$$

$$v = \sqrt{G m' / r} \quad \text{orbiting speed}$$

$$\sqrt{\frac{G m_{\oplus}}{R_{\oplus}}} = \sqrt{g R_{\oplus}} \equiv v_1 = 7.9 \text{ km/s}$$

$$T = \frac{2\pi r}{v} = 2\pi \left(\frac{r^3}{G m'} \right)^{1/2} \geq 2\pi \left(\frac{R_{\oplus}^3}{G m'} \right)^{1/2} = 2\pi \sqrt{R_{\oplus} / g} \\ = 84'26''$$

Launched from the surface of the earth

$$\frac{1}{2}mv_L^2 - G\frac{m_\oplus m}{R_\oplus} = \frac{1}{2}mv^2 - G\frac{m_\oplus m}{r} = -G\frac{m_\oplus m}{2r}$$

$$v_L^2 = 2Gm_\oplus \left(\frac{1}{R_\oplus} - \frac{1}{2r} \right) \geq \frac{2Gm_\oplus}{R_\oplus} \left(1 - \frac{R_\oplus}{2R_\oplus} \right)$$
$$= gR_\oplus = v_1^2$$

Escape speed from the earth

$$\frac{1}{2}mv^2 - G\frac{m_{\oplus}m}{R_{\oplus}} = \frac{1}{2}mv_{\infty}^2 - G\frac{m_{\oplus}m}{r_{\infty}}$$

$$v_E = \sqrt{2Gm_{\oplus}/R_{\oplus}} = \sqrt{2}v_1 = 11.2 \text{ km/s}$$

逃逸速度

$$\frac{v_E}{c} \ll 1 \quad \text{Weak gravitation}$$

$$\frac{v_E}{c} \leq 1 \quad \text{Strong gravitation}$$

Laplacian black hole

gravitational radius

If $v_E = \sqrt{2Gm/R} \geq c$ or $R \leq \frac{2Gm}{c^2} \equiv r_g$

even light can not escape from it. Such a celestial body is called Laplacian black hole.

It is necessary for a black hole to have higher compactness

致密性

$$\frac{m}{R} \geq \frac{c^2}{2G}$$

rather than higher density.

The condition for density is

$$\rho = \frac{m}{\frac{4}{3}\pi R^3} \geq \frac{m}{\frac{4}{3}\pi r_g^3} = \frac{3c^6}{32\pi G^3 m^2} \sim \frac{1}{m^2}$$

A galaxy of $10^9 m_\odot$, its density is as low as $\rho \sim 20 \text{ kg/m}^3$ can still be blackhole.

Neutron stars have very high density,

$$\rho \sim 10^{17} \sim 10^{18} \text{ kg/m}^3$$

they are not the candidates for black holes.

*4.3 Gravitational mass, redshift and collapse

$$m_g \sim W \quad m_i \sim 1/a \quad m_i/m_g = 1 ?$$

$$1. \quad G \frac{m_{\oplus} m_g}{R_{\oplus}^2} = m_i a, \quad a = G \frac{m_{\oplus}}{R_{\oplus}^2} = g$$

$$2. \quad -P = m_g g - m_i \ddot{r}$$

Newton method: simple pendulum

$$m_i L \ddot{\theta} = -m_g g \sin \theta$$

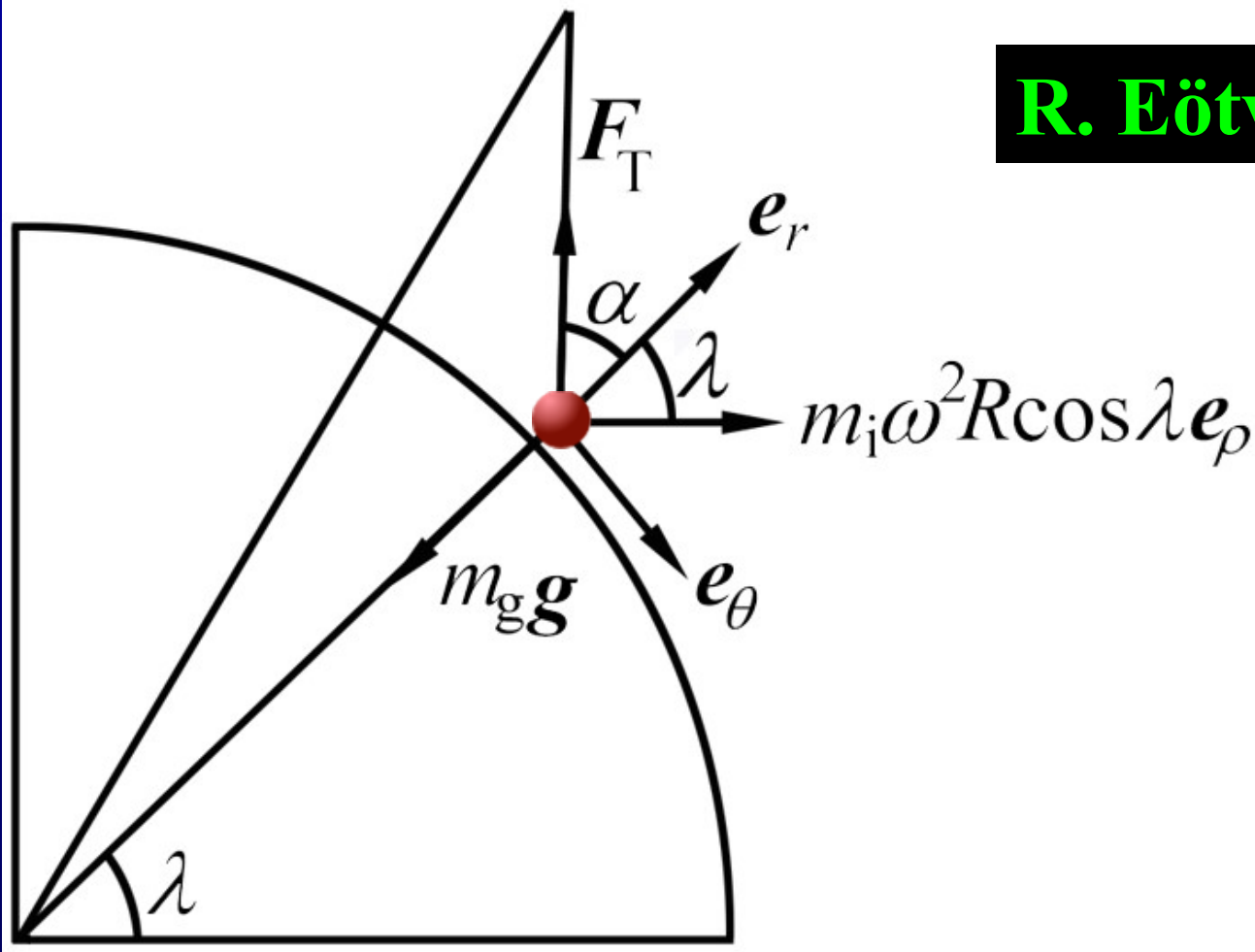
$$\ddot{\theta} + \frac{m_g g}{m_i L} \theta = 0$$

$$\omega^2 = \frac{m_g}{m_i} \frac{g}{L}$$

R. Eötvös 1890+25

‘Oat-Voesh’
厄缶

<https://forvo.com>



$$e_{\theta} : -F_T \sin \alpha + m_i \omega^2 R \cos \lambda \sin \lambda = 0$$

$$e_r : F_T \cos \alpha - m_g g + m_i \omega^2 R \cos^2 \lambda = 0$$

$$\tan \alpha \approx \alpha \approx \frac{m_i \omega^2 R \sin(2\lambda)}{2m_g g} \sim \frac{m_i}{m_g}$$

R. Eötvös 1890+25

$$\frac{m_i}{m_g} = \frac{m_{iPt}}{m_{gPt}}$$

$$5 \times 10^{-8}$$

1964 R. H. Dicke et al

$$10^{-11}$$

1971 Braginski

$$10^{-12}$$

Gravitational red shift

photon's energy (section 22.2) $E = h\nu$

mass-energy relation (section 9.2) $E = mc^2$

Mass of photon $m = h\nu/c^2$

Conservation of energy

$$-h d\nu = mg dz = \frac{h\nu}{c^2} g dz$$

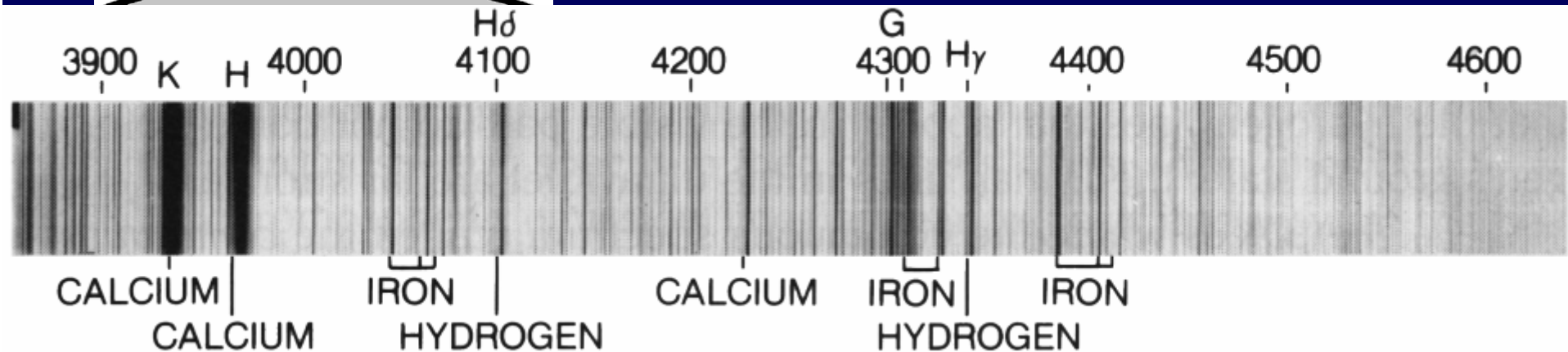
$$\int_{\nu_e}^{\nu_o} \frac{d\nu}{\nu} = - \int_0^H \frac{g}{c^2} dz$$

$$\nu_o = \nu_e \exp\left(-\frac{gH}{c^2}\right) \approx \nu_e \left(1 - \frac{gH}{c^2}\right)$$

1050 P. V. Pound and G. A. Rebka

$$\frac{\Delta \nu}{\nu} = \frac{gH}{c^2} = 2.46 \times 10^{-15}$$

$$(2.57 \pm 0.26) \times 10^{-15}$$





Jefferson laboratory at Harvard

$$-h d\nu = \frac{Gm'm}{(R+z)^2} dz = \frac{h\nu}{c^2} \frac{Gm'}{(R+z)^2} dz$$

$$\int_{\nu_e}^{\nu_o} \frac{d\nu}{\nu} = -G \frac{m'}{c^2} \int_0^H \frac{dz}{(R+z)^2} = -\frac{Gm'}{c^2 R} \frac{H}{R+H}$$

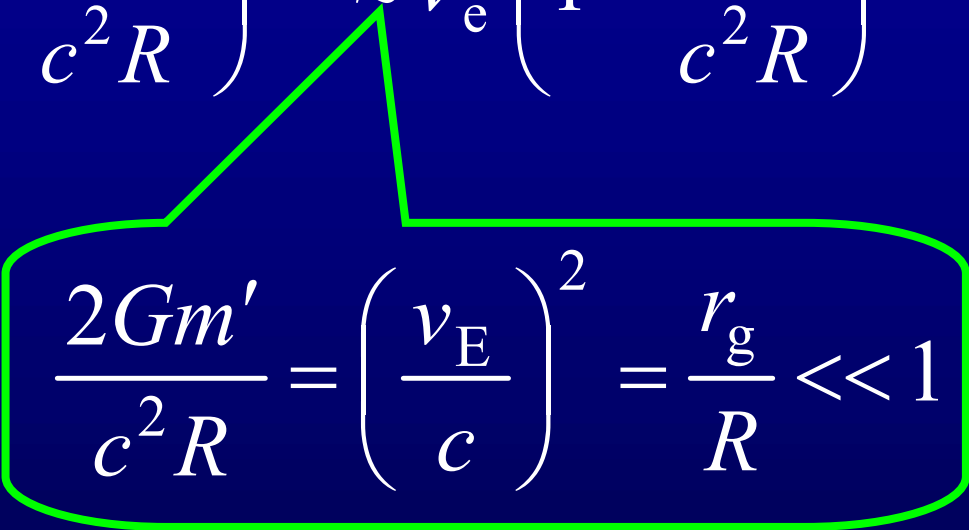
For $H \gg R$

$$\nu_o = \nu_e \exp\left(-\frac{Gm'}{c^2 R}\right) \approx \nu_e \left(1 - \frac{Gm'}{c^2 R}\right)$$

$$dU = -\mathbf{F} \cdot d\mathbf{r} = \frac{Gm'm}{(R+z)^2} dz$$

$$\nu_o = \nu_e \exp\left(-\frac{Gm'}{c^2 R}\right) \approx \nu_e \left(1 - \frac{Gm'}{c^2 R}\right)$$

$$\nu_o = \nu_e \left(1 - \frac{2Gm'}{c^2 R}\right)^{1/2} \approx \nu_e \left(1 - \frac{Gm'}{c^2 R}\right) \quad (\text{GTR})$$



$$\frac{2Gm'}{c^2 R} = \left(\frac{v_E}{c}\right)^2 = \frac{r_g}{R} \ll 1$$

$$\nu_o = \nu_e \exp\left(-\frac{Gm'}{c^2 R}\right) \approx \nu_e \left(1 - \frac{Gm'}{c^2 R}\right)$$

$$\nu_o = \nu_e \left(1 - \frac{2Gm'}{c^2 R}\right)^{1/2} \approx \nu_e \left(1 - \frac{Gm'}{c^2 R}\right) \quad (\mathbf{GTR})$$

$$= \nu_e \left(1 - \frac{r_g}{R}\right)^{1/2} \xrightarrow{R \rightarrow r_g} 0 \quad \mathbf{blackhole}$$

White dwarf: $\frac{\Delta \nu}{\nu} = -5.9 \times 10^{-5} \text{ (calculated)}$

$$\frac{\Delta \nu}{\nu} = -6.6 \times 10^{-5} \text{ (observed)}$$

Gravitational collapse

a cold, diffuse cloud of dust or hydrogen atoms

$$-\Delta U = \Delta E_k + \Delta E_{\text{rad}}$$

contracts $10^3 \text{ K} \longrightarrow 10^7 \text{ K}$ nuclear fusion

outward pressure halts the contraction

hydrogen fuel exhausts $\longrightarrow$ collapses

helium “red giant”

carbon, oxygen,... iron ^{56}Fe

Assignment: 4.4, 4.5, 4.6

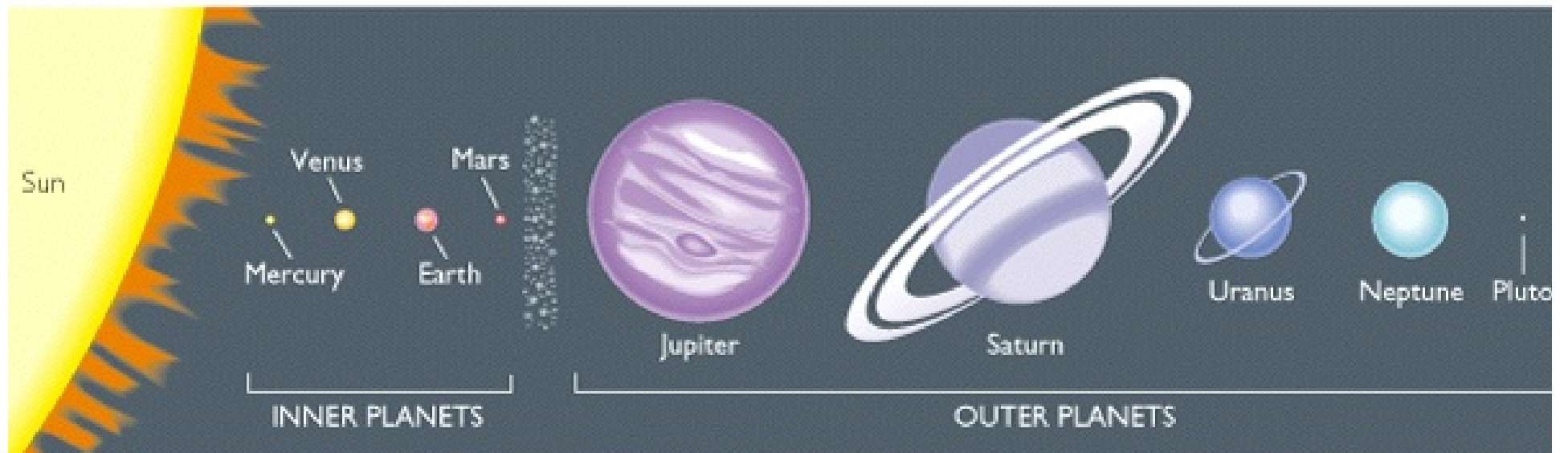
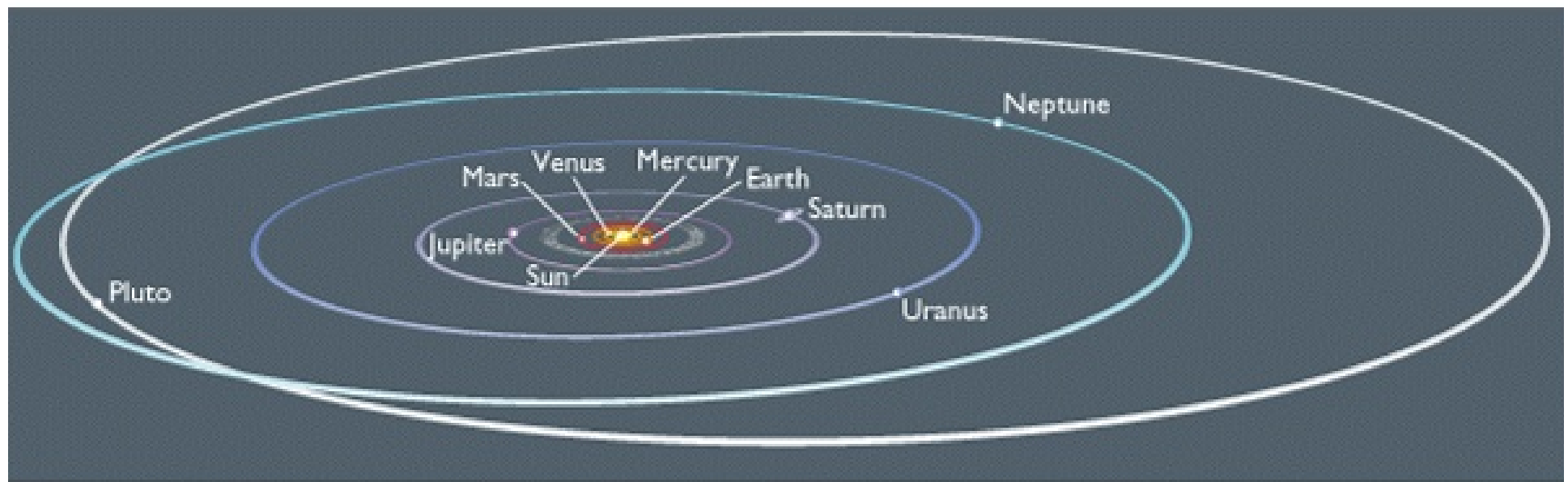
The Pound–Rebka experiment is a well known experiment to test Albert Einstein's theory of general relativity. It was proposed by Robert Pound and his graduate student Glen A. Rebka Jr. in 1959, and was the last of the classical tests of general relativity to be verified (in the same year). It is a gravitational redshift experiment, which measures the redshift of light moving in a gravitational field, or, equivalently, a test of the general relativity prediction that clocks should run at different rates at different places in a gravitational field. It is considered to be the experiment that ushered in an era of precision tests of general relativity.

The test was carried out at Harvard University's Jefferson laboratory. A solid sample containing iron (^{57}Fe) emitting gamma rays was placed in the center of a loudspeaker cone which was placed near the roof of the building. Another sample containing ^{57}Fe was placed in the basement. The distance between this source and absorber was 22.5 meters (73.8 ft). The gamma rays traveled through a Mylar bag filled with helium to minimize scattering of the gamma rays. A scintillation counter was placed below the receiving ^{57}Fe sample to detect the gamma rays that were not absorbed by the receiving sample. By vibrating the speaker cone the gamma ray source moved with varying speed, thus creating varying Doppler shifts. When the Doppler shift canceled out the gravitational blueshift, the receiving sample absorbed gamma rays and the number of gamma rays detected by the scintillation counter dropped accordingly. The variation in absorption could be correlated with the phase of the speaker vibration, hence with the speed of the emitting sample and therefore the Doppler shift. To compensate for possible systematic errors, Pound and Rebka varied the speaker frequency between 10 Hz and 50 Hz, interchanged the source and absorber-detector, and used different speakers (ferroelectric and moving coil magnetic transducer). The reason for exchanging the positions of the absorber and the detector is doubling the effect. Pound subtracted two experimental results:

- (1) the frequency shift with the source at the top of the tower
- (2) the frequency shift with the source at the bottom of the tower

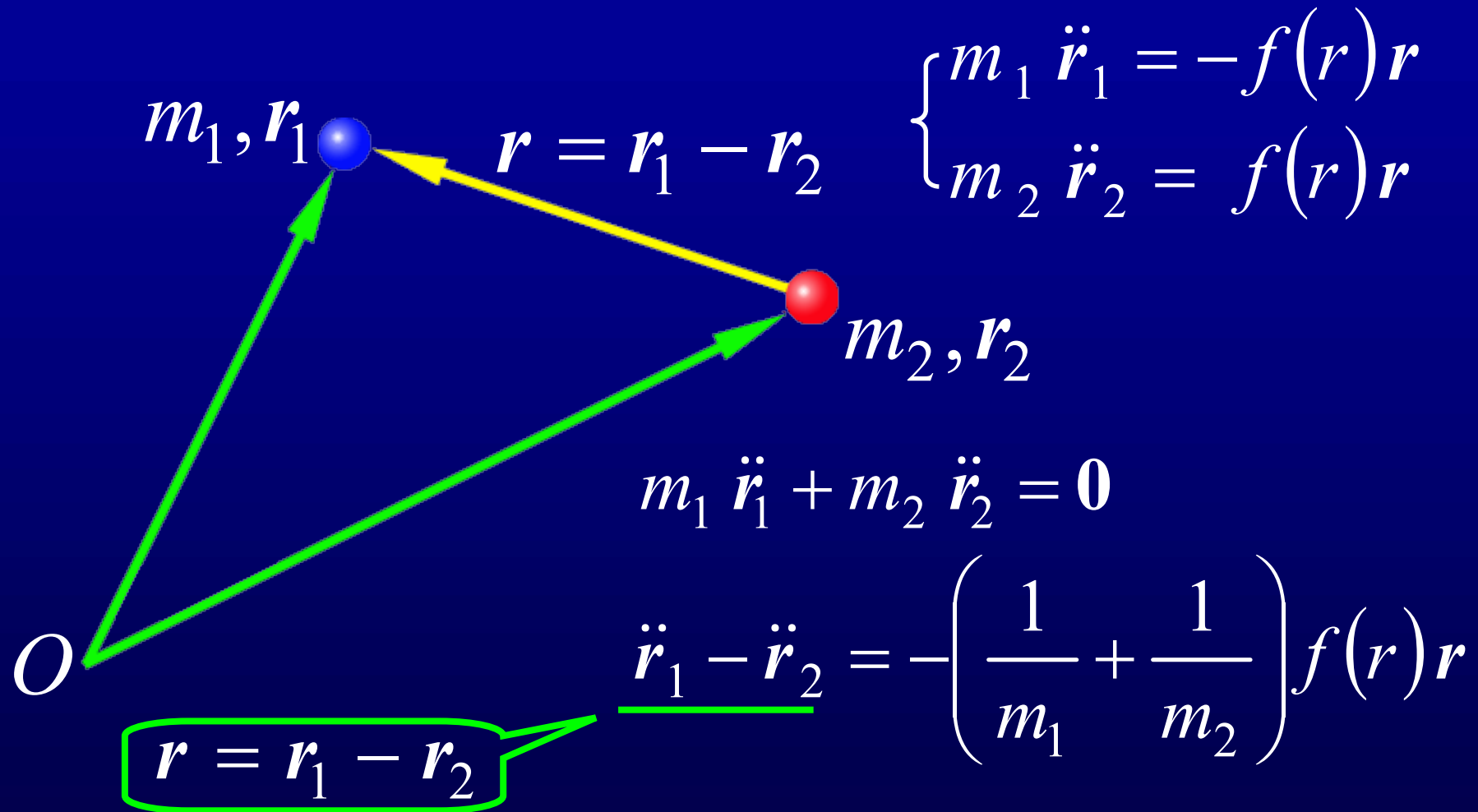
The frequency shift for the two cases has the same magnitude but opposing signs. When subtracting the results, Pound and Rebka obtained a result twice as big as for the one-way experiment.

The result confirmed that the predictions of general relativity were borne out at the 10% level.[3] This was later improved to better than the 1% level by Pound and Snider.



4.4 Kepler problem and *scattering

$$f \sim G \frac{m_1 m_2}{r^3}$$



1

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{0}$$

$$\ddot{\mathbf{r}} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r) \mathbf{r}$$

Define

$$m_1 + m_2 \equiv m_C$$

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \equiv m_C \mathbf{r}_C$$

$$\frac{1}{m_1} + \frac{1}{m_2} \equiv \frac{1}{\mu}$$

reduced mass

*Which particle's eq?
mass?
Position vector?
force?*

$$\ddot{\mathbf{r}}_C = 0$$

$$\mu \ddot{\mathbf{r}} = -f(r) \mathbf{r}$$

$$\boxed{2} \quad \mathbf{M} = \mathbf{r} \times f(r) \mathbf{r} = \mathbf{0}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{C} \quad \text{Conservation of } \mathbf{L}$$

- $\mathbf{r}$ and $\mathbf{p}$ are co-planar
- $\mathbf{L} = \mathbf{r} \times \mu \left(\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta \right) = \mu r^2 \dot{\theta} \mathbf{k}$
- areal “velocity”

For central force (有心力场) $\mathbf{F} = F(r) \hat{\mathbf{r}}$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F(r) \hat{\mathbf{r}} = \mathbf{0},$$

the torque exerted about the center is zero.

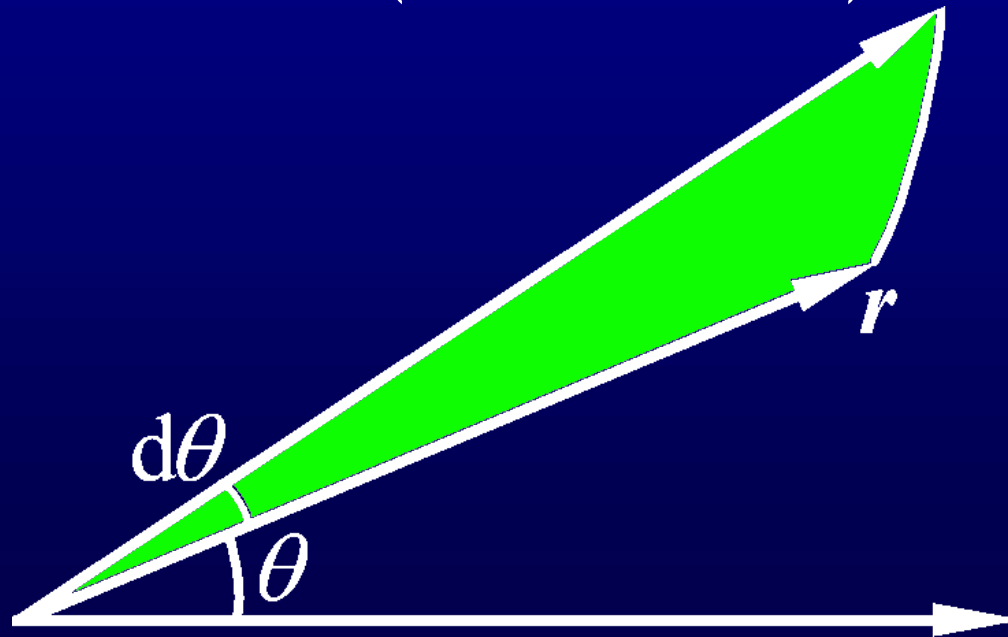
$$\boxed{2} \quad \mathbf{M} = \mathbf{r} \times f(r) \mathbf{r} = \mathbf{0}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{C}$$

Conservation of $\mathbf{L}$

• $\mathbf{r}$ and $\mathbf{p}$ are co-planar

$$\bullet \mathbf{L} = \mathbf{r} \times \mu (\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta) = \mu r^2 \dot{\theta} \mathbf{k}$$



$$\frac{1}{2} r d\theta \cdot r$$

$$\boxed{2} \quad \mathbf{M} = \mathbf{r} \times f(r) \mathbf{r} = \mathbf{0}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{C}$$

Conservation of $\mathbf{L}$

- $\mathbf{r}$ and $\mathbf{p}$ are co-planar
- $\mathbf{L} = \mathbf{r} \times \mu (\dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta) = \mu r^2 \dot{\theta} \mathbf{k}$
- areal “velocity”

$$\frac{dA}{dt} = \frac{1}{2} \frac{r d\theta}{dt} r = \frac{L}{2\mu} = \text{constant}$$

- $\dot{\theta}$ can never change sign.

3**1st integrals**

$$\begin{aligned}\mu \left(\ddot{r} - r \dot{\theta}^2 \right) &= -F(r), \\ \mu \left(r \ddot{\theta} + 2\dot{r} \dot{\theta} \right) &= 0 \longrightarrow \mu r^2 \dot{\theta} = \text{constant } \mathbf{L}\end{aligned}$$

$$\dot{r} \cdot dr = \frac{d\dot{r}}{dt} \cdot \frac{dr}{dt} dt = d\dot{r} \cdot \dot{r} = d\left(\frac{1}{2} \dot{r}^2\right)$$

$$-r \dot{\theta}^2 \cdot dr = -r \left(\frac{L}{\mu r^2} \right)^2 \cdot dr = -\frac{L^2}{\mu^2 r^3} dr = d\left(\frac{L^2}{2\mu^2 r^2} \right)$$

$$[-f(r)r] \cdot dr = F \cdot dr = -dU = -d\left(-G \frac{m_1 m_2}{r}\right) \equiv -d\left(-\frac{k}{r}\right)$$

$$\underbrace{\frac{1}{2} \mu \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)}_{v^2} - \frac{k}{r} = E$$

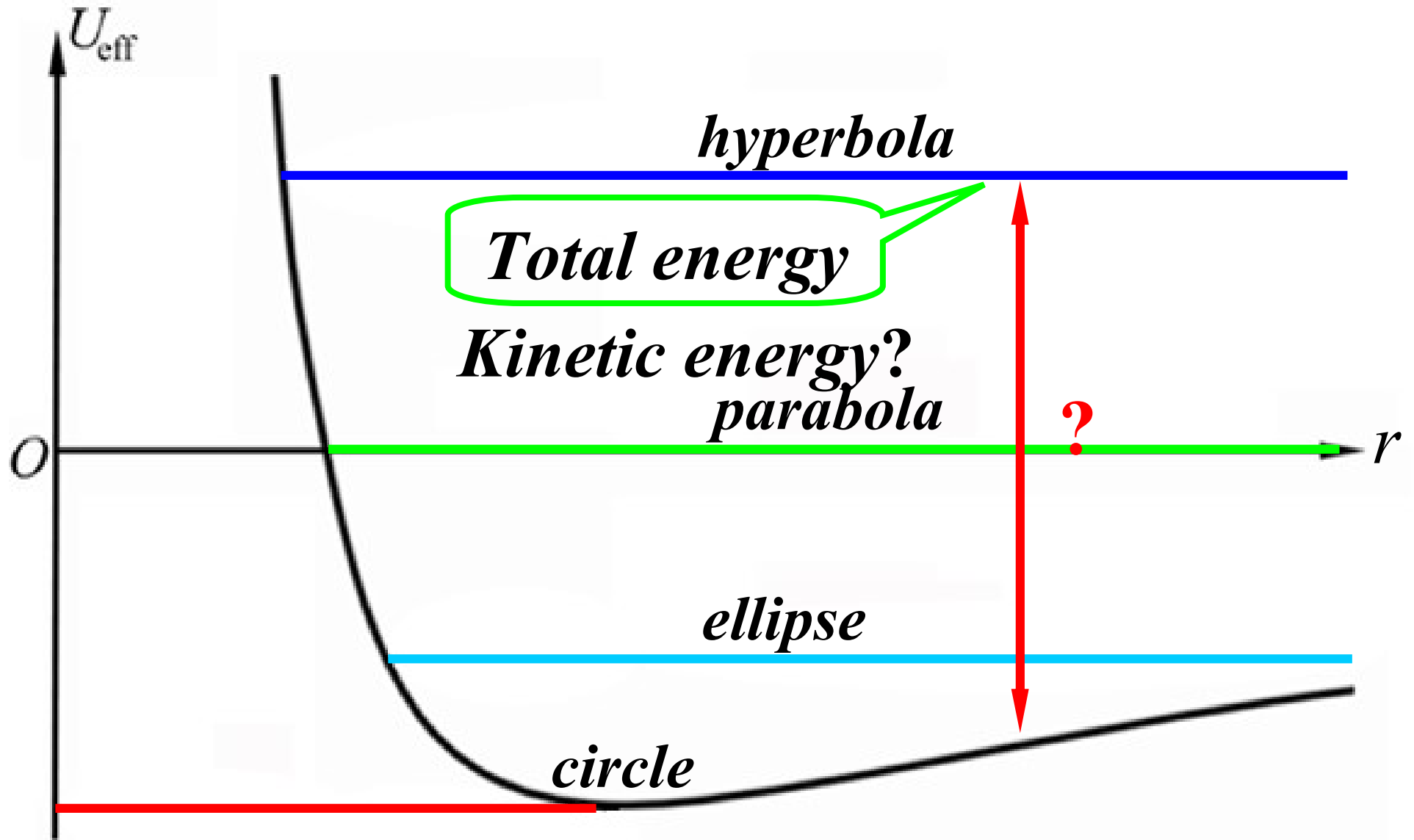
4 Effective potential

$$\frac{1}{2} \mu r^2 \dot{\theta}^2$$

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + U(r) \equiv \frac{1}{2} \mu \dot{r}^2 + U_{\text{eff}}(r)$$

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

$$\frac{L^2}{2\mu r^2} - \frac{k}{r} = \begin{cases} ? & r \rightarrow 0 \\ ? & r \rightarrow \infty \end{cases}$$



5

$$r(t) \longrightarrow r = r(\theta)$$

$$E = \frac{1}{2} \mu \dot{r}^2 + \boxed{\frac{L^2}{2\mu r^2} + U(r)}$$

$$\dot{r} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{L}{\mu r^2} \boxed{\frac{dr}{d\theta}}$$

$$\begin{aligned}
d\theta &= \frac{L}{\mu r^2 \left\{ \frac{2}{\mu} \left[E - \frac{L^2}{2\mu r^2} - U(r) \right] \right\}^{1/2}} dr \quad \dot{r} \\
&= \frac{L}{\mu r^2 \left\{ \frac{2}{\mu} \left[E - \frac{L^2}{2\mu r^2} + \frac{k}{r} \right] \right\}^{1/2}} dr \\
&= \frac{L}{\mu r^2 \left\{ \frac{1}{\mu^2} \left[2\mu E - \frac{L^2}{r^2} + \frac{2\mu k}{r} \right] \right\}^{1/2}} dr
\end{aligned}$$

$$= -\frac{L}{\left[2\mu E + \frac{\mu^2 k^2}{L^2} - \left(\frac{L^2}{r^2} - \frac{2\mu k}{r} + \frac{\mu^2 k^2}{L^2}\right)\right]^{1/2}} d\left(\frac{1}{r}\right)$$

$$= -\frac{L}{\left[2\mu E + \frac{\mu^2 k^2}{L^2} - \left(\frac{L}{r} - \frac{\mu k}{L}\right)^2\right]^{1/2}} d\left(\frac{1}{r}\right)$$

$$\sim -\frac{dx}{\sqrt{a^2 - x^2}} = d\left[\arccos\left(\frac{x}{a}\right)\right]$$

$$\theta = \arccos \frac{\frac{L}{r} - \frac{\mu k}{L}}{\left\{ \frac{\mu^2 k^2}{L^2} + 2\mu E \right\}^{1/2}} + \theta_0$$


$$\cos(\theta - \theta_0) = \frac{\frac{p}{r} - 1}{\sqrt{1 + 2\mu E \frac{L^2}{\mu^2 k^2}}} \equiv \frac{\frac{p}{r} - 1}{e}$$

$$p = \frac{L^2}{\mu k}$$

$$r = \frac{p}{1 + e \cos(\theta - \theta_0)}$$

$$E = -\left(1 - e^2\right) \frac{k}{2p}$$

$$r = \frac{p}{1 + e \cos(\theta - \theta_0)}$$

eccentricity	energy	orbit
$e = 0$	$E = -\frac{k}{2p}$	circle
$0 < e < 1$	$-\frac{k}{2p} < E < -\frac{k}{2a} < 0$	ellipse
$e = 1$	$E = 0$	parabola
$e > 1$	$E > 0$	hyperbola

For ellipse orbit

$$p = \frac{L^2}{\mu k}$$

$$r_{\min} \equiv \frac{p}{1+e} \leq r \leq \frac{p}{1-e} \equiv r_{\max}$$

$$a = \frac{r_{\min} + r_{\max}}{2} = \frac{p}{1-e^2}$$

$$b = \sqrt{ap}$$

$$A = \int \frac{dA}{dt} dt = \frac{L}{2\mu} t$$

$$T = \frac{2\mu}{L} (\pi a b) = \frac{2\mu\pi}{L} a^{3/2} \sqrt{p}$$

$$= \frac{2\mu\pi}{L} a^{3/2} \sqrt{\frac{L^2}{\mu G m' m}} = 2\pi a^{3/2} \sqrt{\frac{m' m}{(m' + m)} \frac{1}{G m' m}}$$

$$T = 2\pi a^{3/2} \sqrt{\frac{1}{G(m' + m)}} \approx 2\pi a^{3/2} \sqrt{\frac{1}{G m'}} \text{ (Kepler)}$$

***Hyperbola orbit**

$$E = \frac{1}{2} \mu v_0^2$$

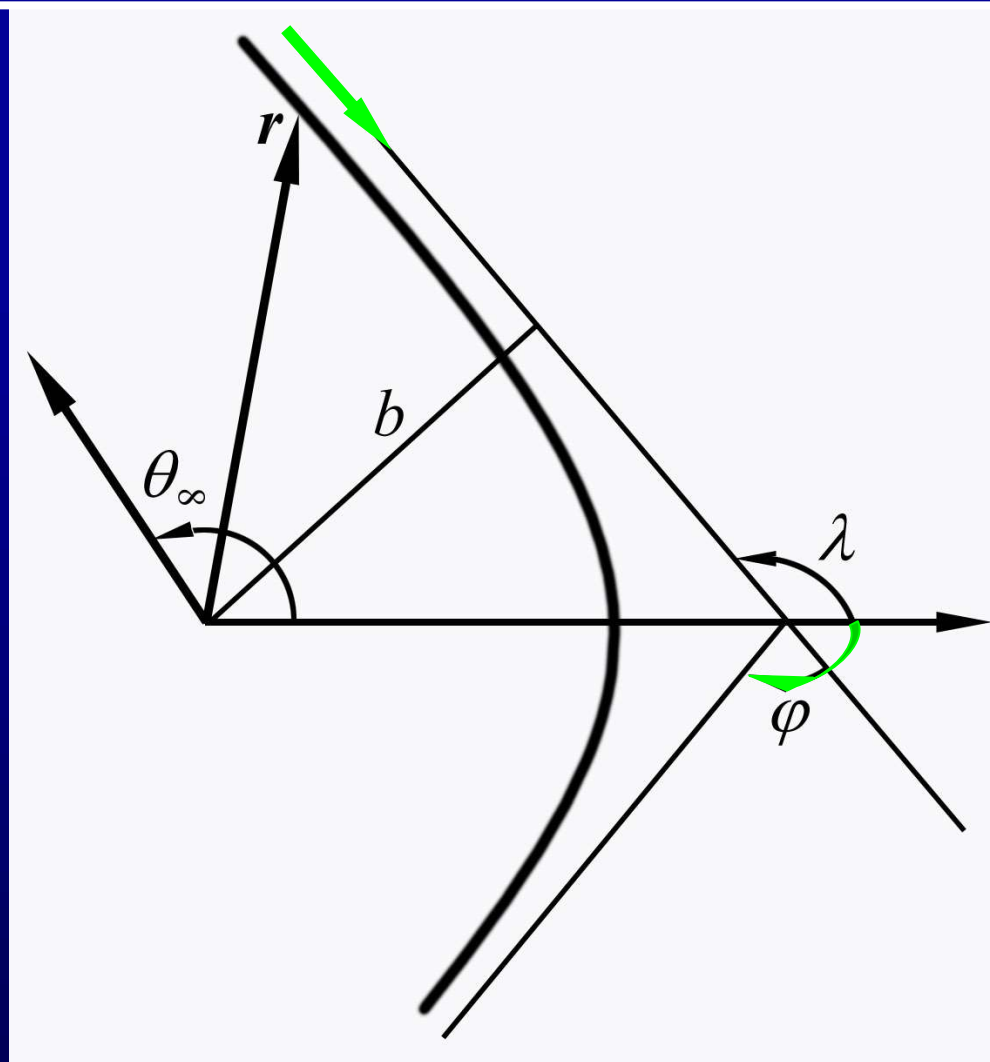
$$L = \mu v_0 b$$

Scattering angle

$$\varphi = 2\lambda - \pi \approx 2\theta_\infty - \pi$$

$$r = \frac{p}{1 + e \cos \theta}$$

$$\cos \theta_\infty = -1/e$$



$$\tan \frac{\varphi}{2} = -\cot \theta_{\infty} = \frac{k}{2Eb}$$

$$-\cot \theta_{\infty} = \frac{-\cos \theta_{\infty}}{\sin \theta_{\infty}} = \frac{\frac{1}{e}}{\sqrt{1 - \frac{1}{e^2}}} = \frac{1}{\sqrt{e^2 - 1}}$$

$$= \frac{1}{\sqrt{2\mu E \frac{L^2}{\mu^2 k^2}}} = \frac{k}{\sqrt{2E \frac{\mu^2 v_0^2 b^2}{\mu}}}$$

Eq.(4.4.17)

2E

Assignment: 4.7, 4.12, *4.14

Another Way to Solve the Kepler Problem

In the planar polar coordinates (r, θ) , where r is the radial coordinate and θ is the polar angle, the equation of motion

$$\mu \ddot{\mathbf{r}} = -F(r) \hat{\mathbf{r}}, \quad F(r) = \frac{Gm_1 m_2}{r^2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \mathbf{e}_r,$$

can be written in the form of components

$$\mathbf{e}_r : \mu (\ddot{r} - r \dot{\theta}^2) = -F(r), \quad \mathbf{e}_\theta : \mu (r \ddot{\theta} + 2\dot{r} \dot{\theta}) = 0. \quad (1)$$

The second equation can be rewritten as

$$\mu r (r \ddot{\theta} + 2\dot{r} \dot{\theta}) = \frac{d}{dt} (\mu r^2 \dot{\theta}) = 0,$$

and it can be integrated to give

$$\mu r^2 \dot{\theta} = L = \text{Const.} \quad (2)$$

where L is constant angular momentum about the Sun.

Elimination $\dot{\theta}$ in the first equation of (1) gives

$$\ddot{r} - \frac{L^2}{\mu^2} \frac{1}{r^3} = -\frac{F(r)}{\mu}. \quad (3)$$

It is a second order nonlinear differential equation. By substituting $u = 1/r$ into (3), noting that

$$\dot{r} = \frac{d}{dt} \left(\frac{1}{u} \right) = -\frac{1}{u^2} \frac{du}{dt} = -r^2 \frac{du}{d\theta} \dot{\theta} = -\frac{L}{\mu} \frac{du}{d\theta},$$

$$\begin{aligned} \ddot{r} &= \frac{d}{dt} (\dot{r}) = \frac{d}{dt} \left(-\frac{L}{\mu} \frac{du}{d\theta} \right) = -\frac{L}{\mu} \frac{d^2 u}{d\theta^2} \dot{\theta}, \\ &= -\frac{L^2}{\mu^2 r^2} \frac{d^2 u}{d\theta^2} = -\frac{L^2 u^2}{\mu^2} \frac{d^2 u}{d\theta^2} \end{aligned}$$

Eq.(3) can be rewritten as

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2} \frac{1}{u^2} F\left(\frac{1}{u}\right) \quad (4)$$

Eq.(4) is called the Binet equation which is firstly derived by French mathematician Jacques Philippe Marie Binet.

Substituting

$$F\left(\frac{1}{u}\right) = Gm_1m_2u^2$$

into (4), we obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm_1m_2\mu}{L^2}. \quad (5)$$

Eq.(3) can be rewritten as

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{L^2} \frac{1}{u^2} F\left(\frac{1}{u}\right) \quad (4)$$

Eq.(4) is called the Binet equation which is firstly derived by French mathematician Jacques Philippe Marie Binet.

Substituting

$$F\left(\frac{1}{u}\right) = Gm_1m_2u^2$$

into (4), we obtain

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm_1m_2\mu}{L^2}. \quad (5)$$

Eq. (5) is a second order non-homogenous(非齐次) linear differential equation. Obviously, a particular solution of (5) is

$$u_p = \frac{Gm_1m_2\mu}{L^2}.$$

And the general solution of the corresponding homogenous(齐次) equation is

$$u = C \cos(\theta - \theta_0).$$

Thus, the general solution of (5) is

$$u = C \cos(\theta - \theta_0) + \frac{Gm_1m_2\mu}{L^2}$$

or

$$r = \frac{1}{C \cos(\theta - \theta_0) + \frac{Gm_1m_2\mu}{L^2}} = \frac{\frac{L^2}{Gm_1m_2\mu}}{1 + \frac{CL^2}{Gm_1m_2\mu} \cos(\theta - \theta_0)}. \quad (6)$$

In (6)

$$\frac{L^2}{Gm_1m_2\mu} = p$$

is the semi-latus rectum and

$$\frac{CL^2}{Gm_1m_2\mu} = e$$

is the eccentricity. The total energy of the planet is

$$\begin{aligned}
E &= \frac{1}{2} \mu \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{Gm_1m_2}{r} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{Gm_1m_2}{r} = \frac{L^2}{2\mu} \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] - Gm_1m_2u \\
&= \frac{L^2}{2\mu} \left[C^2 \sin^2(\theta - \theta_0) + \left(C \cos(\theta - \theta_0) + \frac{Gm_1m_2\mu}{L^2} \right)^2 \right] \\
&\quad - Gm_1m_2 \left(C \cos(\theta - \theta_0) + \frac{Gm_1m_2\mu}{L^2} \right) \\
&= \frac{L^2}{2\mu} C^2 - \frac{G^2 m_1^2 m_2^2 \mu}{2L^2} = \frac{G^2 m_1^2 m_2^2 \mu}{2L^2} \left(\frac{L^4}{G^2 m_1^2 m_2^2 \mu^2} - 1 \right) \\
&= \frac{G^2 m_1^2 m_2^2 \mu}{2L^2} (e^2 - 1) = -\frac{Gm_1m_2}{2p} (1 - e^2).
\end{aligned}$$

4.5 Gravitational field

$$\mathbf{F} = -G \frac{m' m}{r^3} \mathbf{r} \equiv m \mathbf{g}_f$$

$$\mathbf{g}_f = \frac{\mathbf{F}}{m} = -\frac{Gm'}{r^3} \mathbf{r} \quad \mathbf{g}_f(\mathbf{r}) = -\sum_i \frac{Gm_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i)$$

Gravitational field exists for certain mass (distribution).

$\mathbf{g}_f$ gravitational field strength

gravitational acceleration

in the inertial frame S

$$\dot{\mathbf{v}} = \mathbf{F}/m = \mathbf{g}_f$$

in the noninertial frame S'

$$\dot{\mathbf{v}}' = \dot{\mathbf{v}} - \dot{\mathbf{u}}$$

Example 4.4

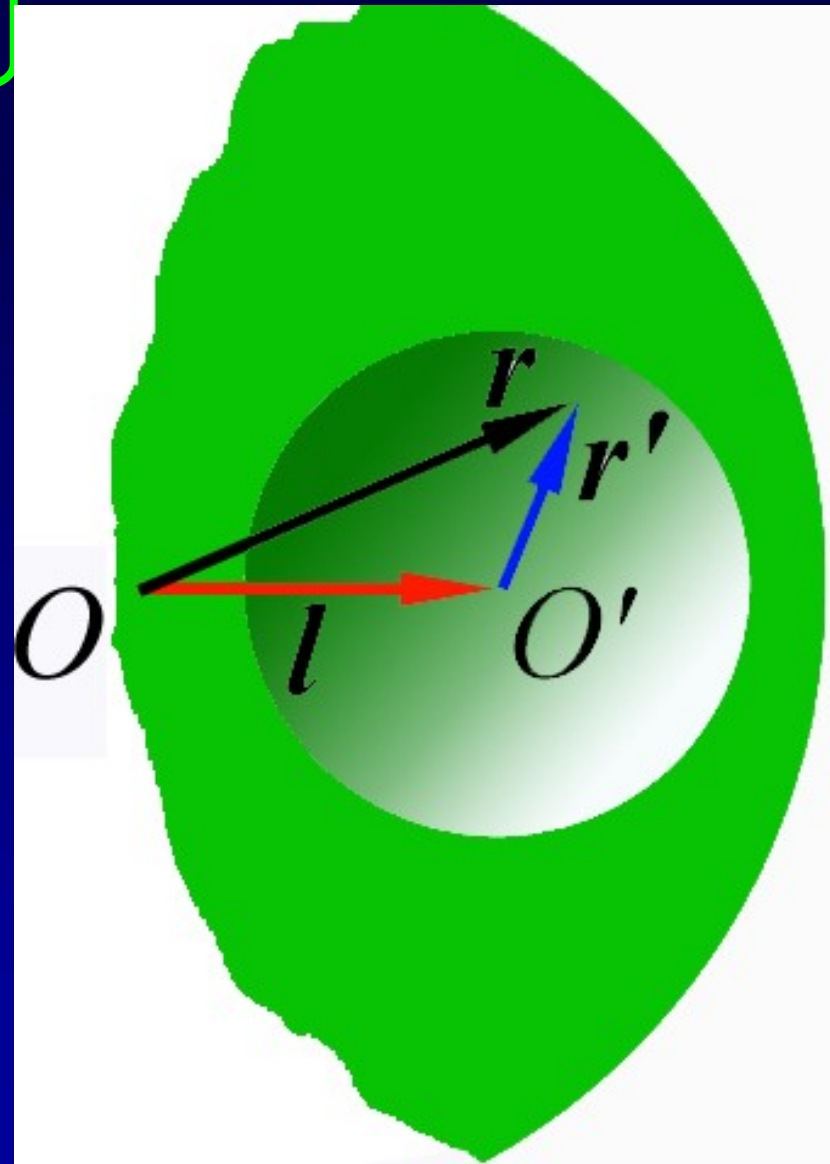
Inside a sphere

$$g(r) = -G \left(\frac{4}{3} \pi r^3 \rho \right) \frac{r}{r^3}$$
$$= -\frac{4}{3} \pi G \rho r$$

In the cavity

$$g = g(r) - g(r')$$
$$= -\frac{4}{3} \pi G \rho l$$

m'



Gravitational potential

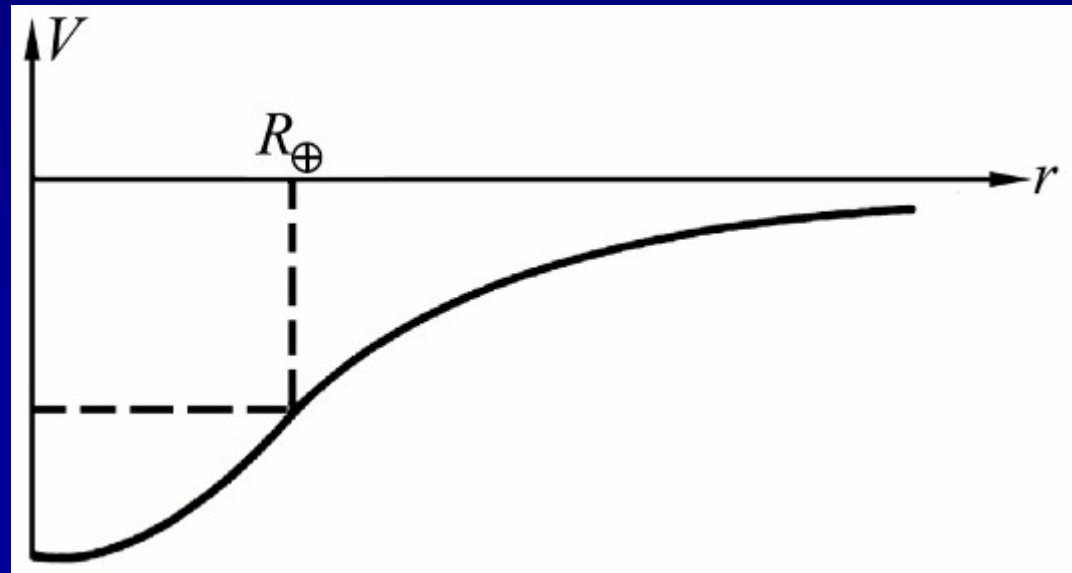
$$\mathbf{F} = -\nabla U \quad \mathbf{g}_f = \frac{\mathbf{F}}{m} = -\nabla \frac{U}{m} \equiv -\nabla V$$

Around particle m'

$$V = -\frac{Gm'}{r} + C$$

Sphere or shell

potential energy?



The gravitational potential around multiple particles is

$$V(\mathbf{r}) = - \sum_i \frac{Gm_i}{|\mathbf{r} - \mathbf{r}_i|}$$

**It is a scalar sum instead of the vector sum for the field.
For continuous distribution of mass we**

$$\mathbf{g}_f(\mathbf{r}) = -G \int \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') d\tau' \quad \text{integral of vector}$$

$$V(\mathbf{r}) = -G \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}') d\tau' \quad \text{integral of scalar}$$

$$\mathbf{g}_f = -\nabla V$$

The gradient of a scalar field

derivative ----rate of change

The gradient measures the rate and direction of the change in a scalar field.

$$V(\mathbf{r}) = -\frac{Gm'}{|\mathbf{r}|} \quad V(x, y, z) = -\frac{Gm'}{\sqrt{x^2 + y^2 + z^2}}$$

In the Cartesian coordinates

$$\text{grad}V(x, y, z) = \frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k} = \nabla V$$

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad \text{del operator or nabla operator}$$

The infinitesimal displacement in the spherical coordinates is

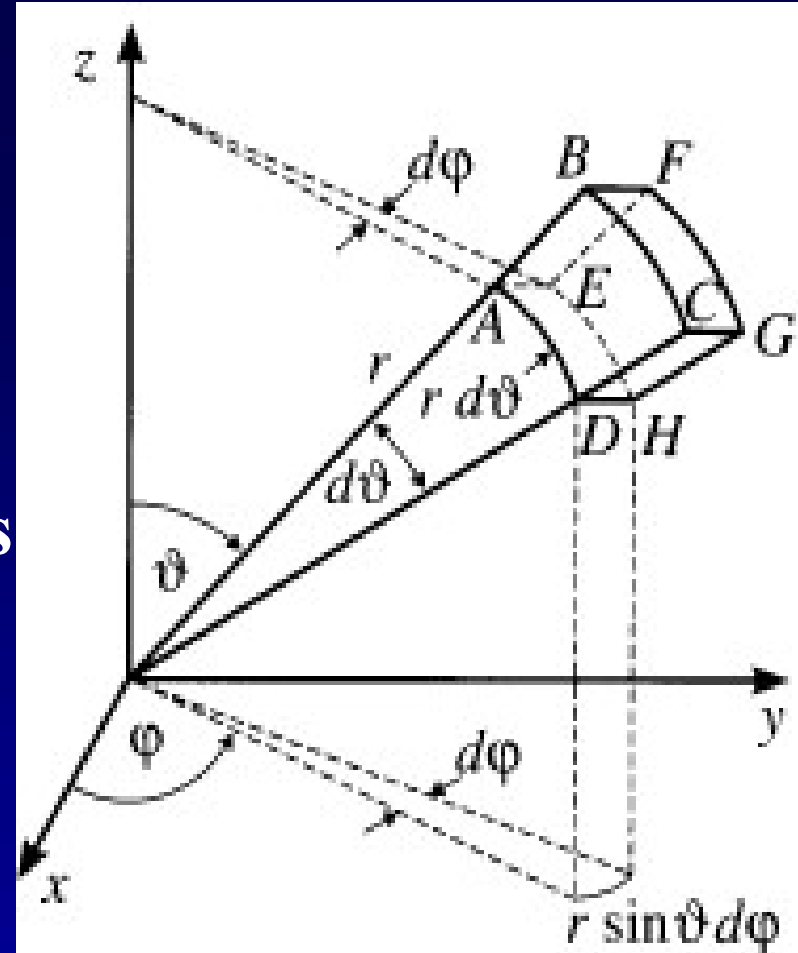
$$d\mathbf{r} = dr\mathbf{e}_r + r d\theta\mathbf{e}_\theta + r \sin \theta d\phi\mathbf{e}_\phi.$$

The 3 infinitesimal line elements are

$$dr, \quad r d\theta \quad \text{and} \quad r \sin \theta d\phi.$$

The volume element is

$$d\tau = r^2 \sin \theta dr d\theta d\phi.$$



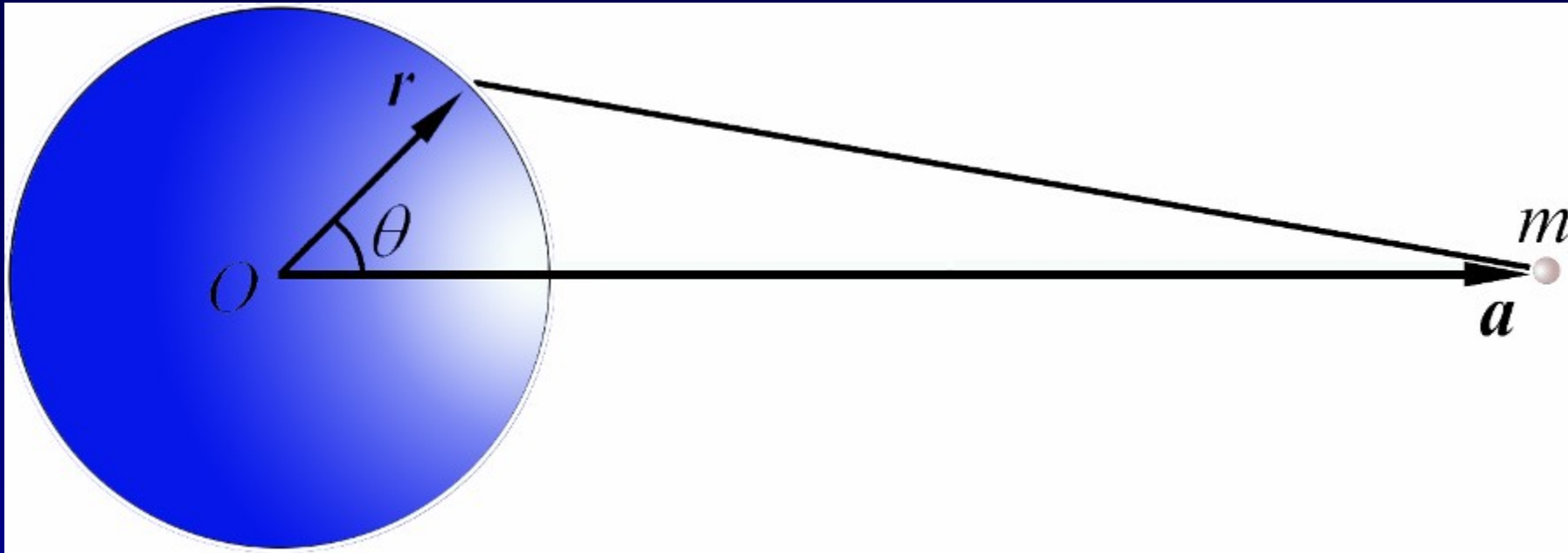
Spherical coordinates: A reminder.

In spherical coordinates

$$V = V(r, \theta, \phi)$$

$$\text{grad} V = \mathbf{e}_r \frac{\partial V}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial V}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi}.$$

Tidal force



Directly find $\mathbf{g}_f(\mathbf{r}) = -\frac{Gm}{|\mathbf{r} - \mathbf{a}|^3}(\mathbf{r} - \mathbf{a})$

or find potential first $V(\mathbf{r}) = -\frac{Gm}{|\mathbf{r} - \mathbf{a}|} = -\frac{Gm}{\sqrt{a^2 + r^2 - 2az}}$

$$\begin{aligned}
V(\boldsymbol{r}) &= -\frac{Gm}{a} \left[1 - 2\frac{z}{a} + \left(\frac{r}{a}\right)^2 \right]^{-1/2} \\
&= -\frac{Gm}{a} \left[1 + \frac{r}{a} \cos \theta + \frac{1}{2} \left(\frac{r}{a}\right)^2 (3 \cos^2 \theta - 1) + \dots \right] \\
&\equiv V_0 + V_1 + V_2 + \dots
\end{aligned}$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$$

$$V(\mathbf{r}) = -\frac{Gm}{a} \left[1 + \frac{r}{a} \cos \theta + \frac{1}{2} \left(\frac{r}{a} \right)^2 (3 \cos^2 \theta - 1) + \dots \right]$$

$$\equiv V_0 + V_1 + V_2 + \dots$$

$$\mathbf{g} = -\nabla V = \frac{Gm}{a^2} \mathbf{k} + g_r \mathbf{e}_r + g_\theta \mathbf{e}_\theta$$

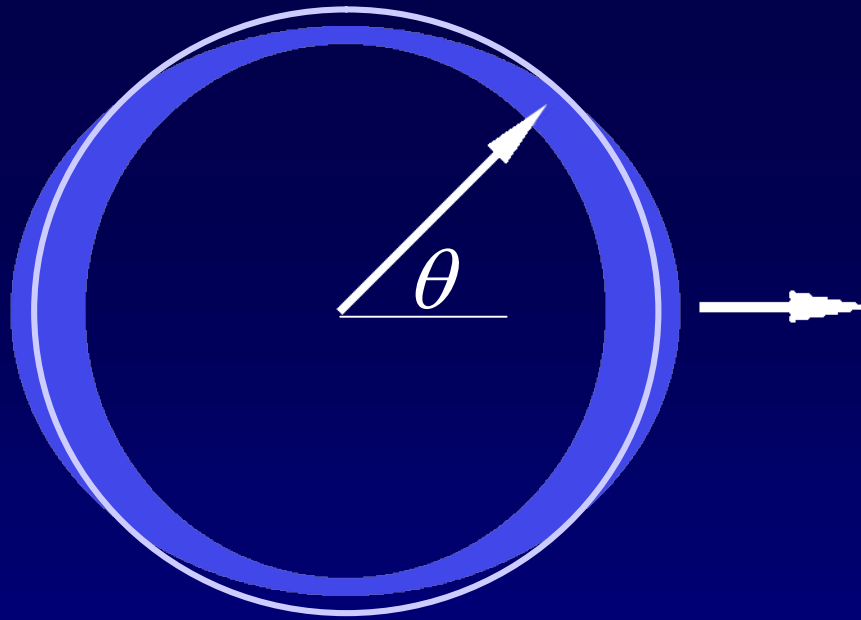
$$-\frac{\partial V_1}{\partial z} \mathbf{k} = \frac{Gm}{a^2} \mathbf{k}$$

$$g_r = -\frac{\partial V_2}{\partial r} = \frac{Gm}{a^2} \frac{r}{a} (3 \cos^2 \theta - 1)$$

$$g_\theta = -\frac{1}{r} \frac{\partial V_2}{\partial \theta} = -\frac{Gm}{a^2} \frac{r}{a} (3 \cos \theta \sin \theta)$$

$$\nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = \mathbf{e}_r \frac{\partial \phi}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi}$$



$$g_r \sim 3 \cos^2 \theta - 1$$

$$g_\theta \sim -3 \cos \theta \sin \theta$$

Roughly in the earth frame (rotating around moon)

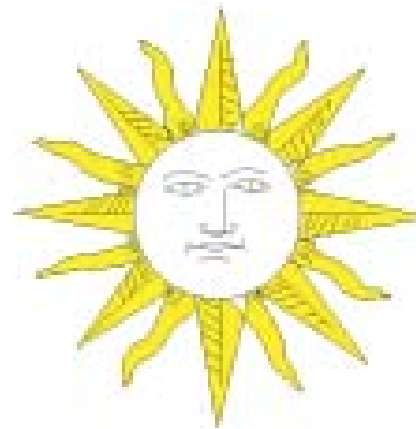
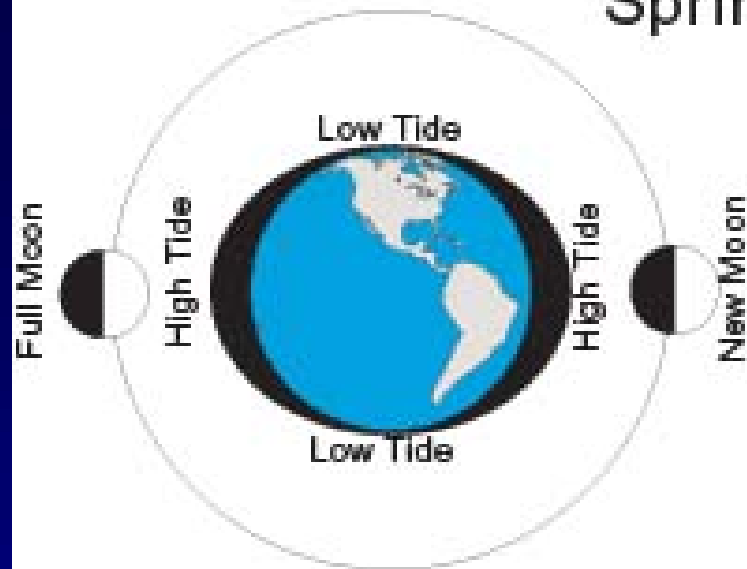
inertial centrifugal force $\sim \Delta m \omega^2 r$

gravitation $\sim \Delta m m' / r^2$

$$\frac{g_m}{g_{\oplus}} \sim \frac{m}{m_{\oplus}} \left(\frac{r}{a} \right)^3 = \left(\frac{7.36 \times 10^{22}}{5.98 \times 10^{24}} \right) \left(\frac{1}{60.3} \right)^3 = 5.60 \times 10^{-8}$$

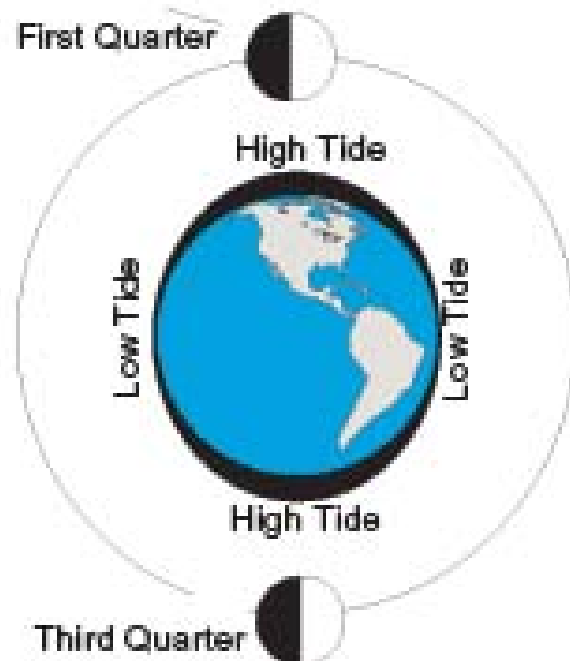
$$\frac{g_{\odot}}{g_{\oplus}} \sim \frac{m_{\odot}}{m_{\oplus}} \left(\frac{r}{a_{\text{SE}}} \right)^3 = \left(\frac{1.99 \times 10^{30}}{5.98 \times 10^{24}} \right) \left(\frac{6.37 \times 10^6}{1.496 \times 10^{11}} \right)^3 = 2.57 \times 10^{-8}$$

Spring Tides

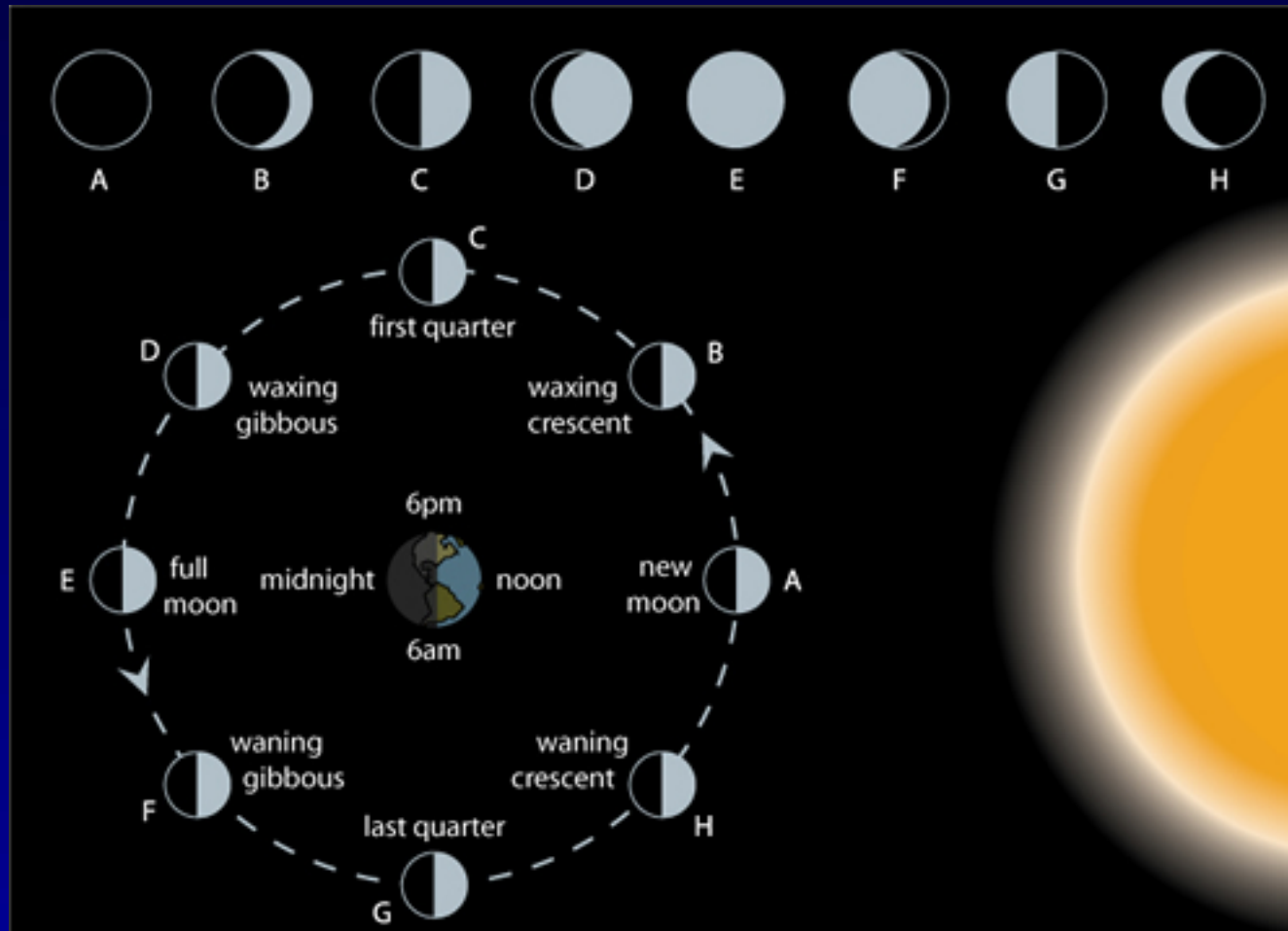


朔望大潮
new moon
朔
full moon
望

Neap Tides



上、下弦
小潮
first or third
quarter of
moon



New Moon - waxing crescent Moon - First Quarter Moon
 - waxing gibbous Moon - Full Moon - waning gibbous
 Moon - Last Quarter Moon - waning crescent Moon -
 New Moon

Earth tides (bodily tides, body tides)

http://en.wikipedia.org/wiki/Earth_tide

中山站恢复重力固体潮观测

<http://www.gzkj.gov.cn/kjxx/newsDetail.jsp?infoId=56910>

Atmospheric tides

http://en.wikipedia.org/wiki/Atmospheric_tide

B. Haurwitz Atmospheric Tides *Science* 19
June 1964 144: 1415-1422

Tidal disruption

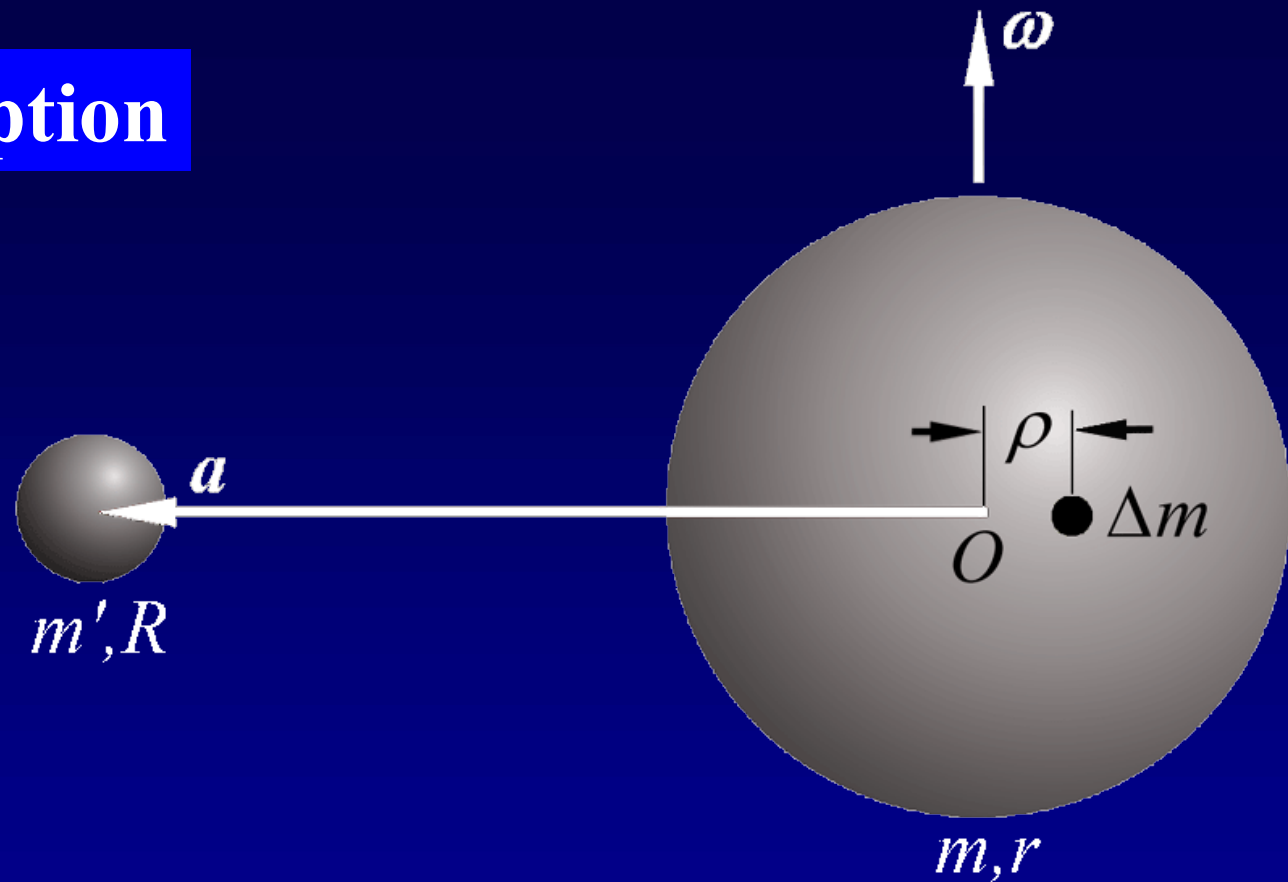
$$3 \cos^2 \theta - 1$$

$$\theta = \pi$$

$$\frac{Gm'}{a^2} \frac{2\rho}{a} \Delta m$$

$$\Delta m \omega^2 \rho$$

$$G \frac{m \Delta m}{r^2} \frac{\rho}{r}$$



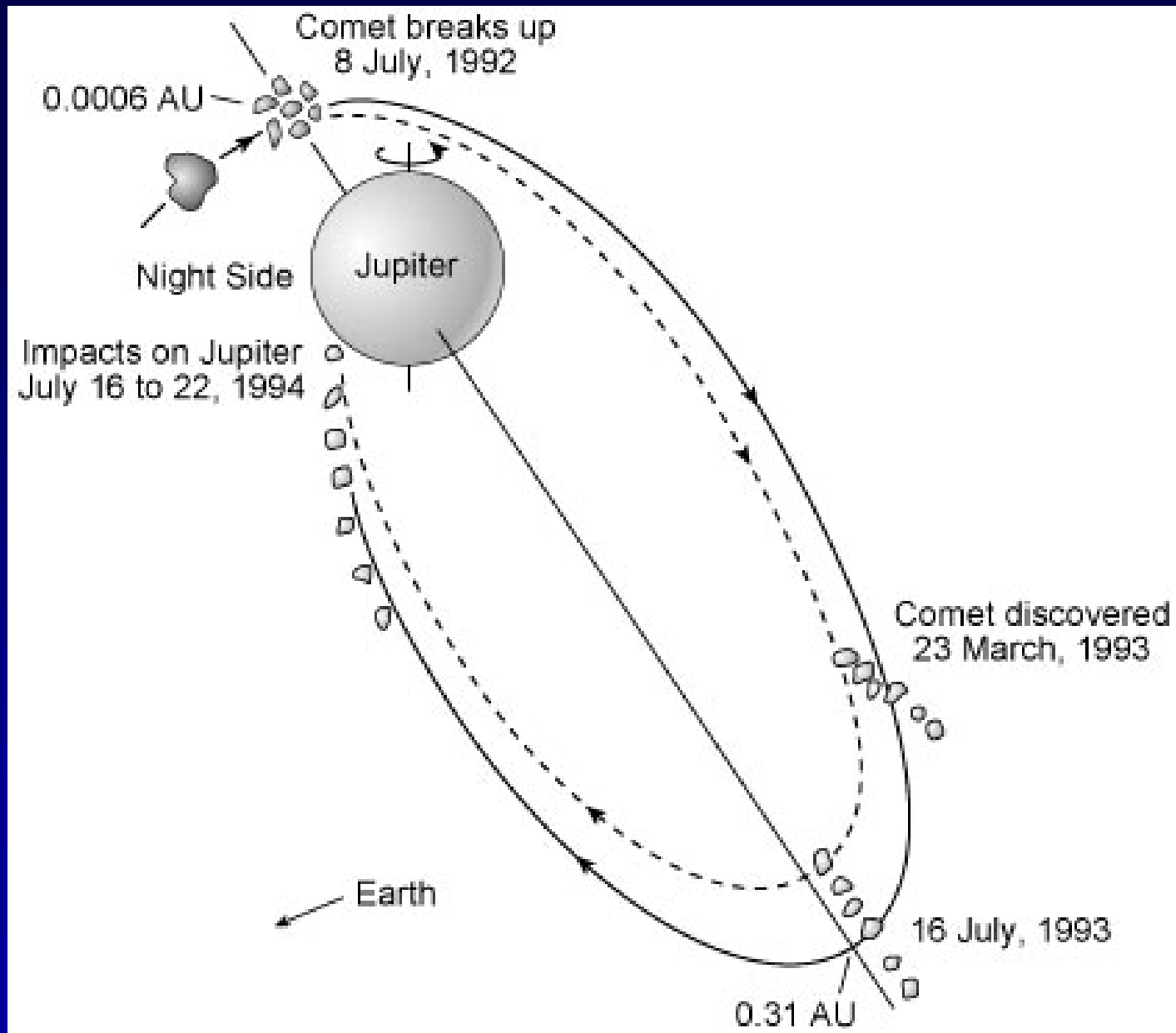
Disruption condition is

$$\rho \Delta m \left(\frac{2Gm'}{a^3} + \omega^2 - \frac{Gm}{r^3} \right) \geq 0$$

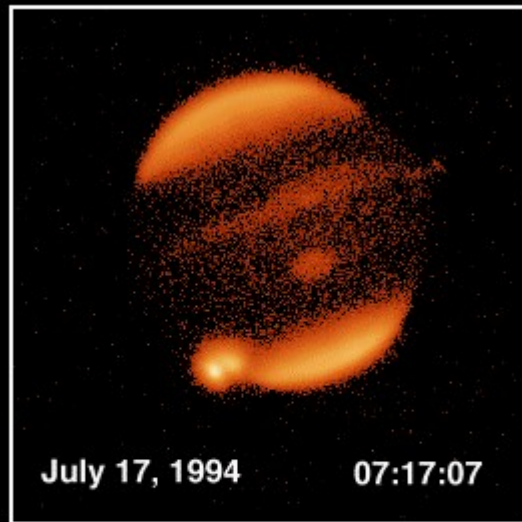
$$\omega = 0 \quad a \leq r \left(\frac{2m'}{m} \right)^{1/3} = 1.26r \left(\frac{m'}{m} \right)^{1/3}$$

$$\omega = \Omega = \frac{\sqrt{G(m' + m)}}{a^{3/2}} \approx \frac{\sqrt{Gm'}}{a^{3/2}} \quad \text{Kepler's 3rd law}$$

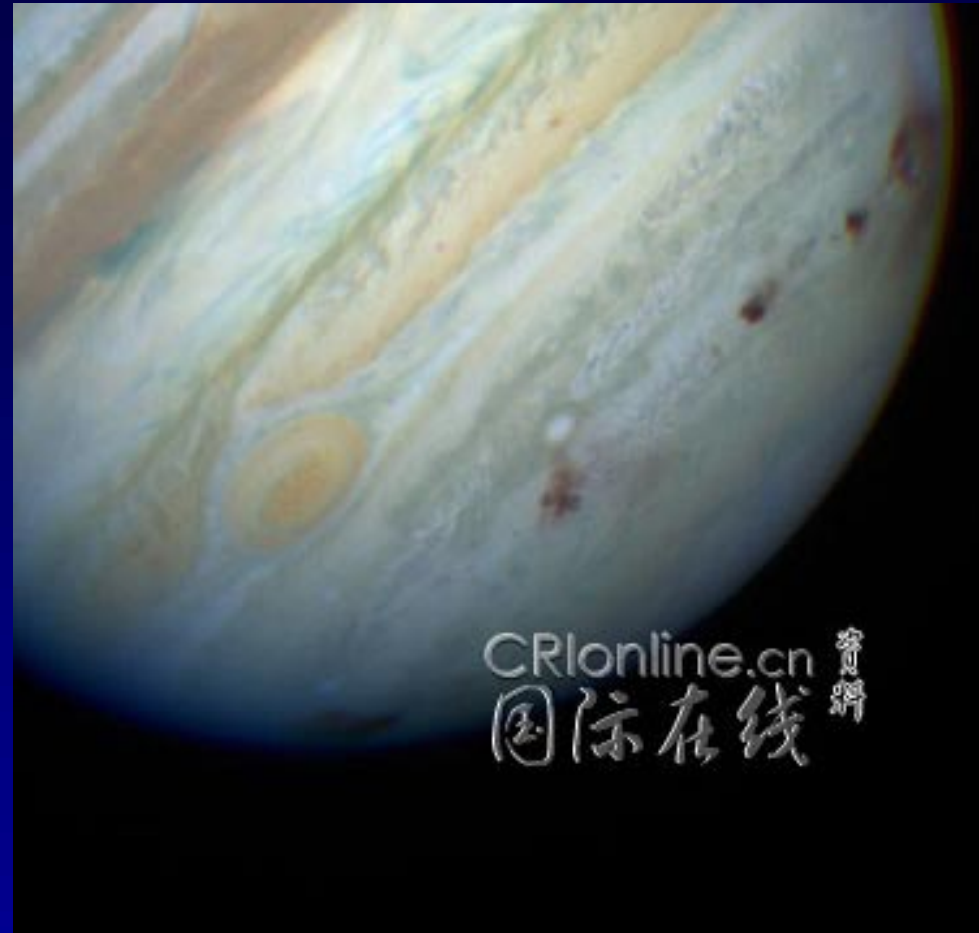
$$a \leq r \left(\frac{3m'}{m} \right)^{1/3} = 1.44r \left(\frac{m'}{m} \right)^{1/3} = 1.44R \left(\frac{\rho_{m'}}{\rho_m} \right)^{1/3}$$



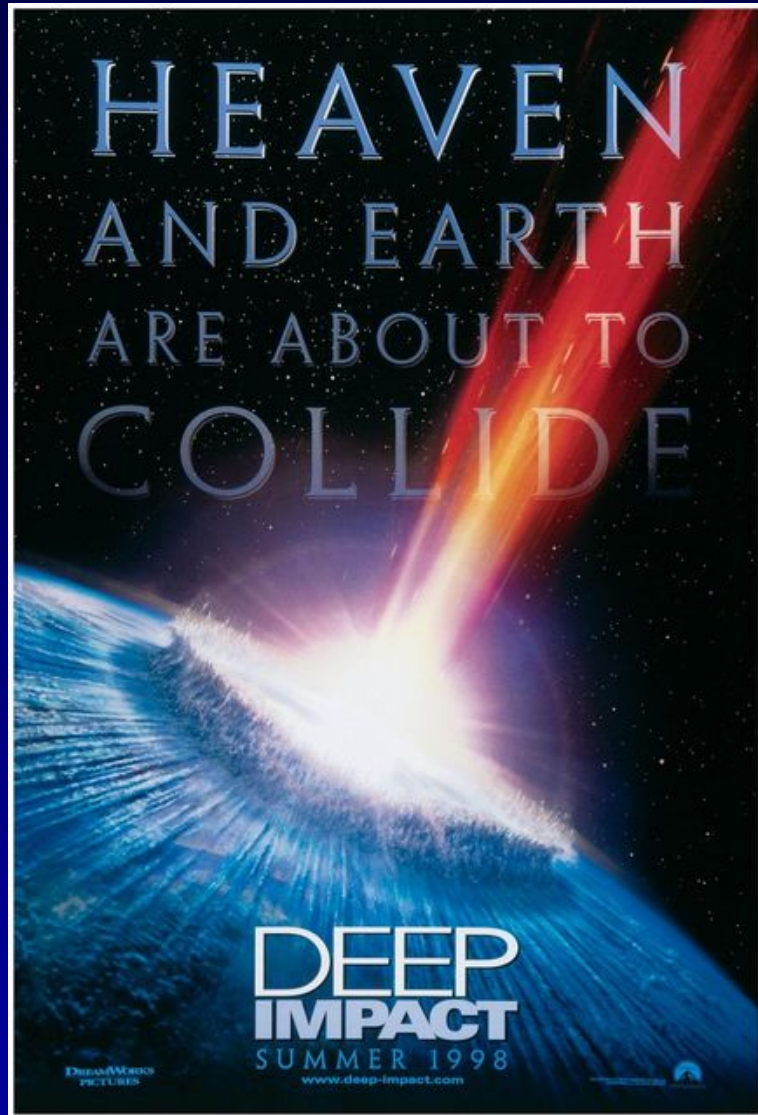
**Shoemaker-Levy 9 striking Jupiter in 1994;
the comet was torn into 21 different pieces
by Jupiter's immense gravity.**



Impact of fragment C of Comet Shoemaker-Levy 9 on Jupiter
(Peter McGregor, ANU 2.3m telescope, CASPIR, 2.34microns)



Black scars reveal the aftermath of Jupiter's impact with nearly two dozen chunks from comet Shoemaker-Levy 9 in 1994. The impacts sent towering mushroom-shaped fireballs of hot gas into the Jovian sky.



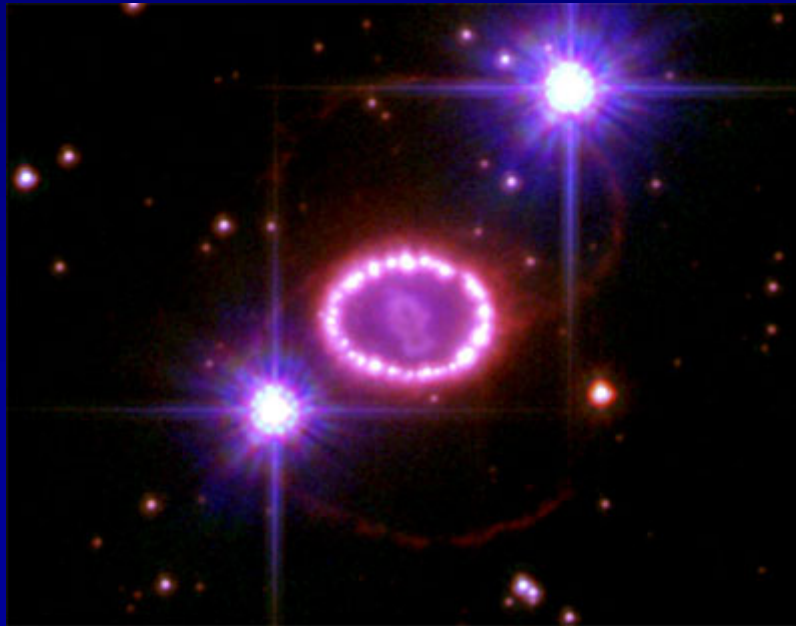
When Marcus Wolf (Charles Martin Smith) is trying to send email about the approaching comet, we see the first few entries in his e-mail inbox. Two of the messages are from "cshoemaker arizona.unv", one of which has the subject line "101 Mir jokes". Carolyn Shoemaker and Eugene Shoemaker are well-known comet experts, credited as "comet advisors" to the movie.

After discovering the comet, one of the astronomers is killed in a car accident. This mirrors the real life car accident death (July 18 1997, in outback Australia) of astronomer Eugene Shoemaker, who helped discover the Shoemaker-Levy 9 Comet.

Gravitational radiation

Gravitational waves are vibrations in the fabric of spacetime which travel at the speed of light and are predicted by Einstein's general theory of relativity.

large masses undergoing high accelerations

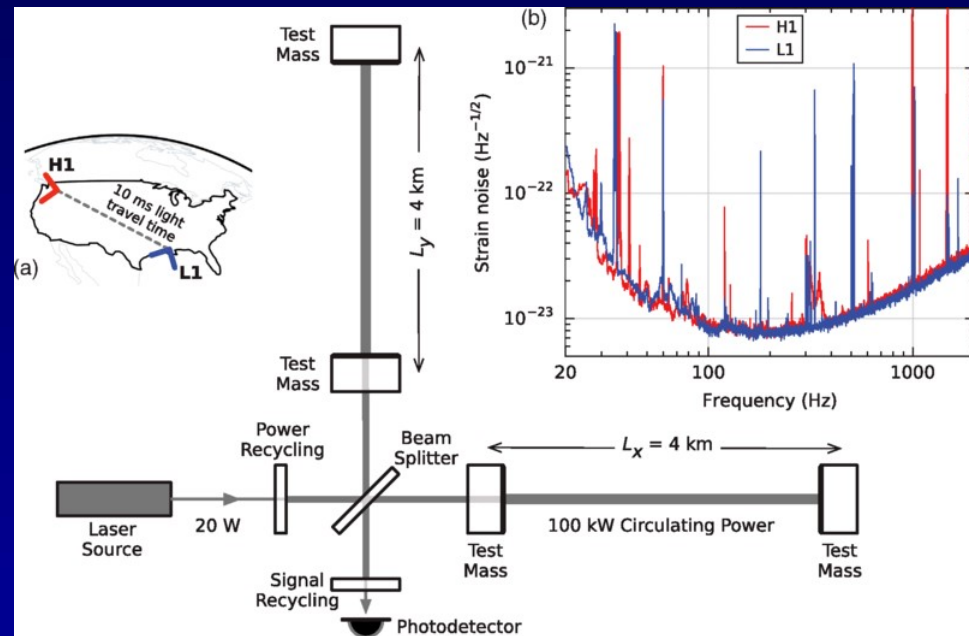


supernova SN 1987A

The Laser Interferometer Gravitational-Wave Observatory (LIGO)

The LIGO Scientific Collaboration (LSC)

On 11 February 2016, the LIGO and Virgo collaborations announced that they succeeded in making the first direct gravitational wave observation on 14 September 2015.



Assignment: 4.18