

大学物理学

University Physics

Syllabus

Instructor :

Wang XU (许 望)

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Expectation:

The course is supposed to be

Bilingual

Interactive

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Interactive

Open

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Bilingual

Interactive

Open

Scientific/Systematic

Scientific way and content of teaching

knowledge

thinking

methodology

morality or ethics

道德

history and forthgoers' innovation

创新

...

pseudo-sciences, pathological science

伪科学、病态科学

The course is supposed to be

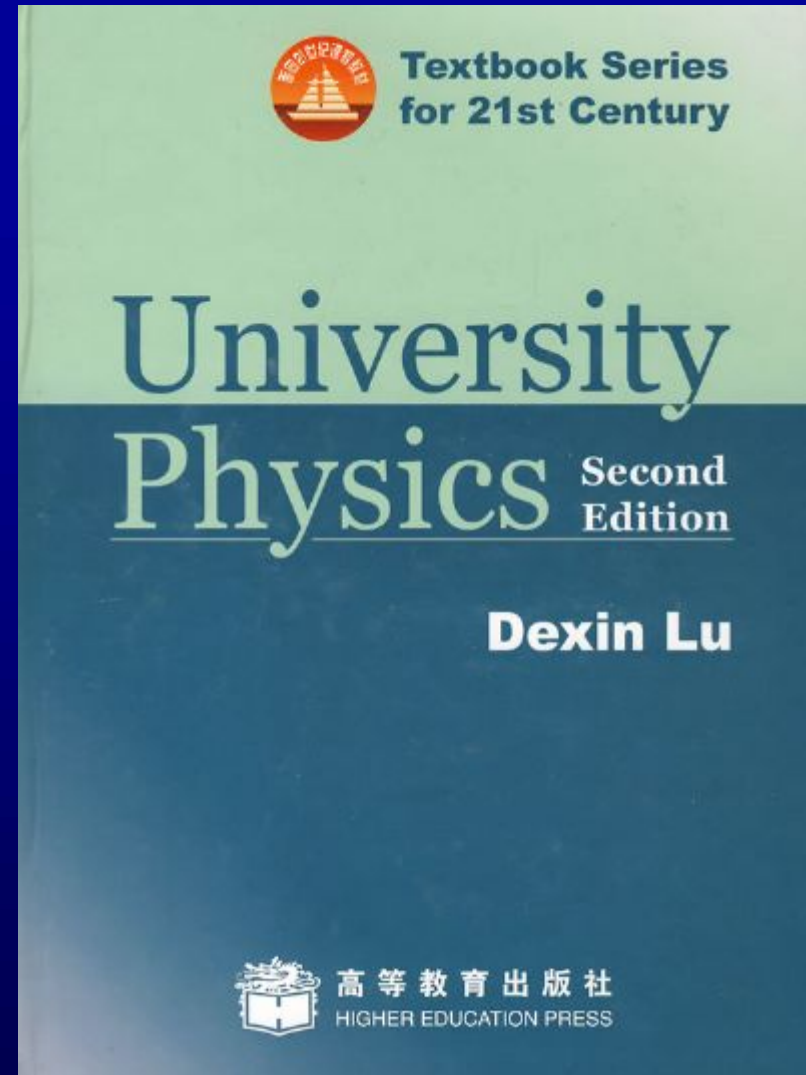
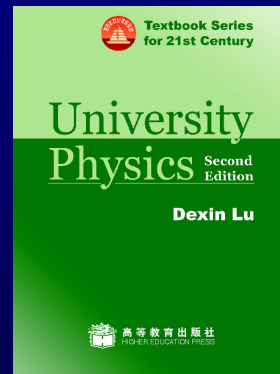
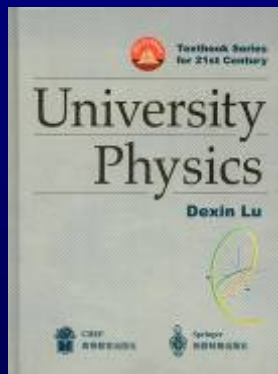
Bilingual

Interactive

Open

Scientific/**S**ystematic

Textbook:



**Use of English version of the
textbook is strongly recommended.**

**You are expected to feel comfortable reading
textbooks in English after this course.**

Grading Policy:

Mid-term Exam	20%
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Final Exam	35%
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Assignments and Attendances (Quizzes*)	30%
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Topic Proposal of the Course Paper	15%
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* take-home quizzes

Due Dates for Assignments:

Thursdays of even-numbered weeks, after two lecture sessions

Due Date for the Topic Proposal:

June 21, Thursday (the last lecture of this semester)

Recommended reference books:

(1) Resnick R, Halliday D, Krane K S. **Physics**
5th ed. John Wiley & Sons, 2002

(2) Ford K W. **Classical and Modern Physics.**
Xerox College Publishing, 1972

(3) Alonso M, Fin E J. **Fundamental University
Physics.** Addison-Wesley Publishing Company,
1978

(4) Orear J. **Physics** MacMillian Publishing Co.
Inc., 1979

中译本： 大学物理学 陈咸亨等译/ 科学出版社

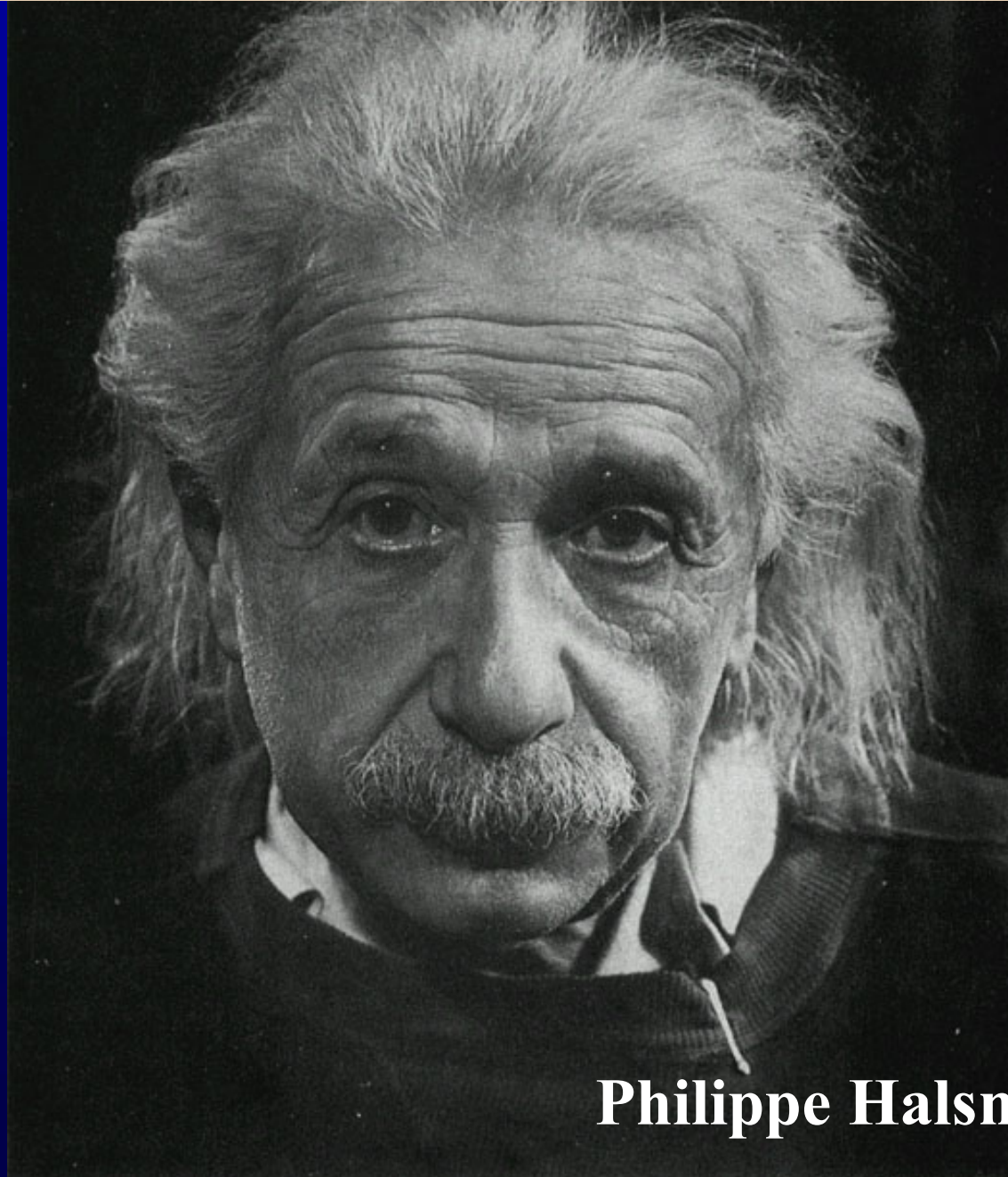
(5) Feynman R P, Leighton R B, and Sands M.
The Feynman Lectures on Physics. Addison-
Wesley Publishing Company, 1975

Public email : uphysicsnju@163.com
password: qwe1rty2

- Lecture notes
- Some references
- Solutions to the quizzes
- Guidelines for the topic proposal of the course paper
- Sample exam papers with complete solutions
- E-book of the textbook
- ...

University Physics

PERSON OF THE CENTURY



Philippe Halsman (1948)



HE WAS AN
ENTHUSIASTIC BUT
NEVER BRILLIANT
AMATEUR MUSICIAN

Introduction

Mechanics

Thermal Physics

Electricity and Magnetism

Fundamental Modern Physics

Chapter 1

Introduction

1.1 What is physics?

The word physics comes from the Greek word *phusika*, from neuter pl. of *phusikos*, meaning “of nature.”

Physics is the discipline of science most directly concerned with the fundamental law of **nature**.

“格物致理” 《大学》

Matter and Motion, and their Interplay
Matter and Interaction

The eternal motive is to seek

Order, symmetry, similarity, simplicity

Matter:

microscopic — fundamental particles

mesoscopic 介观 — nm, cluster 团簇

macroscopic —

cosmological 宇观 — universe

to search for

fractional charge

(B. Fairbank et al 1965, 1977~1981)

dark matter(23%) and dark energy(73%)

antimatter (anti-hydrogen atom found)

AMS

monopole

(Cabrerera et al 1982 **Stanford)**

Motion:

mechanical

thermal

nucleus

particles ...

Interaction:

Einstein: unified field theory

gravitational, weak, electromagnetic, strong

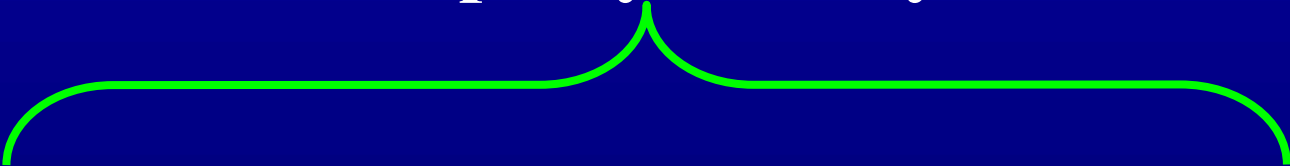
electroweak

“Is There a Fifth Fundamental Force?”



Interaction:

supersymmetry

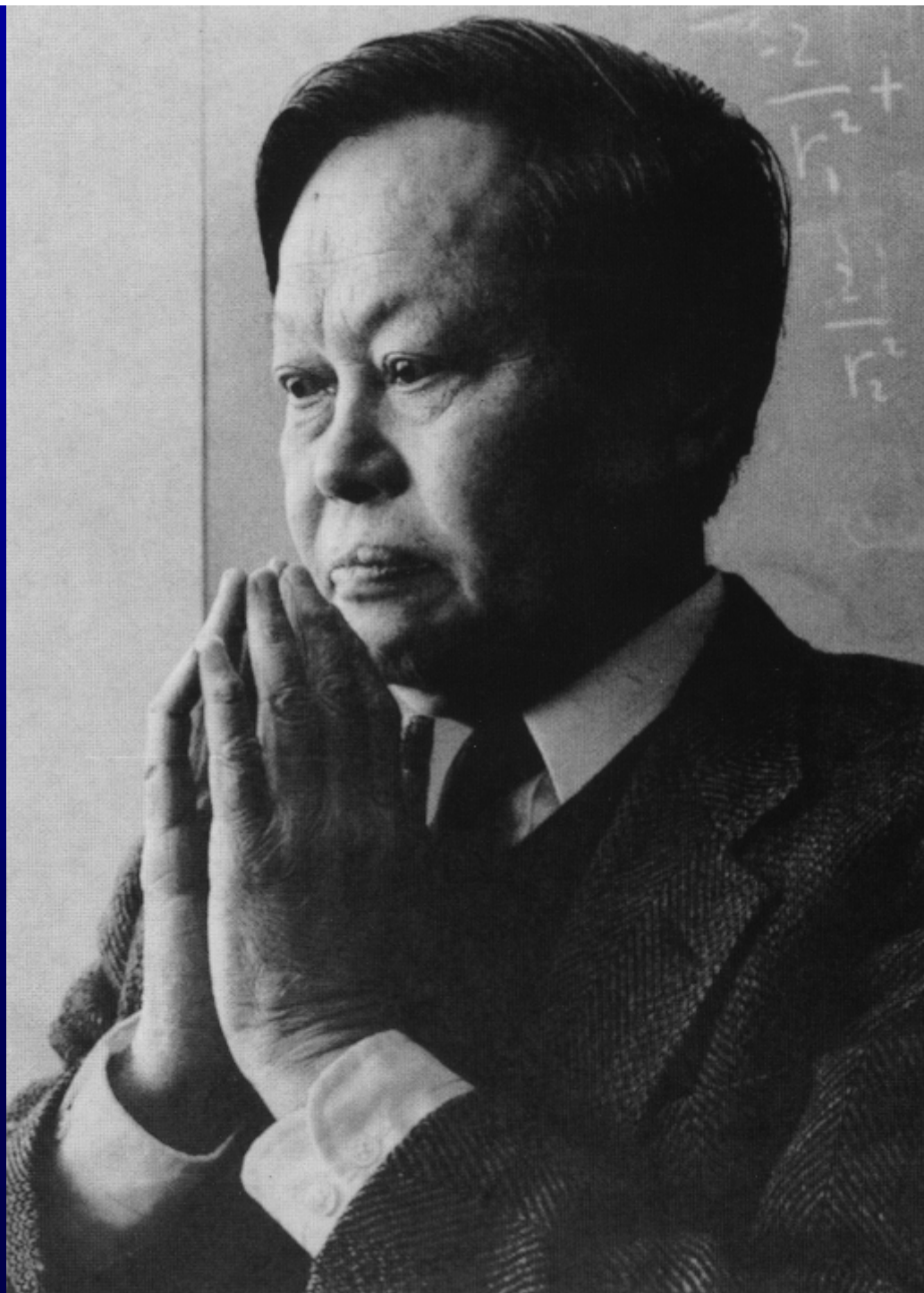


gravitational, weak, electromagnetic, strong



Grand Unification Theory

C.N. Yang

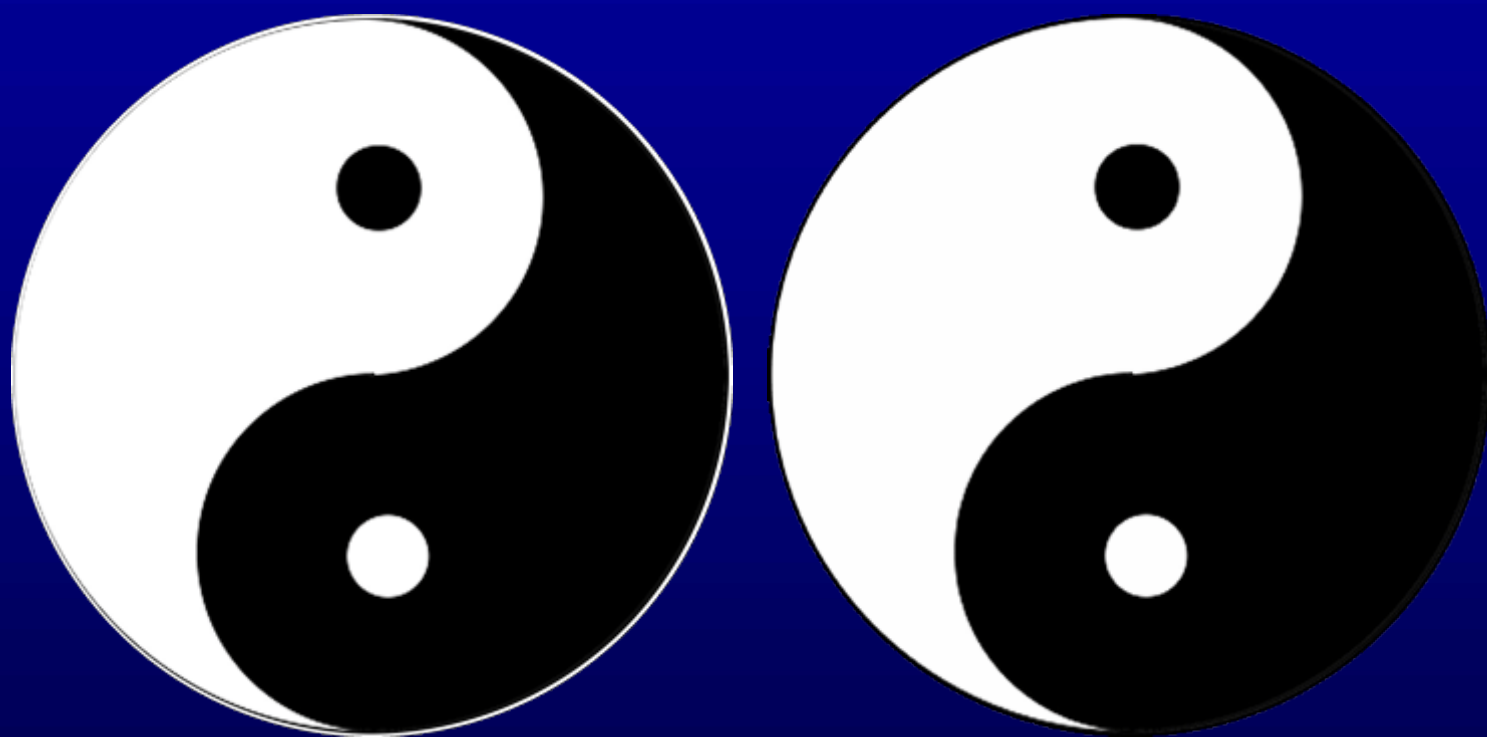


邓伟摄(1994)

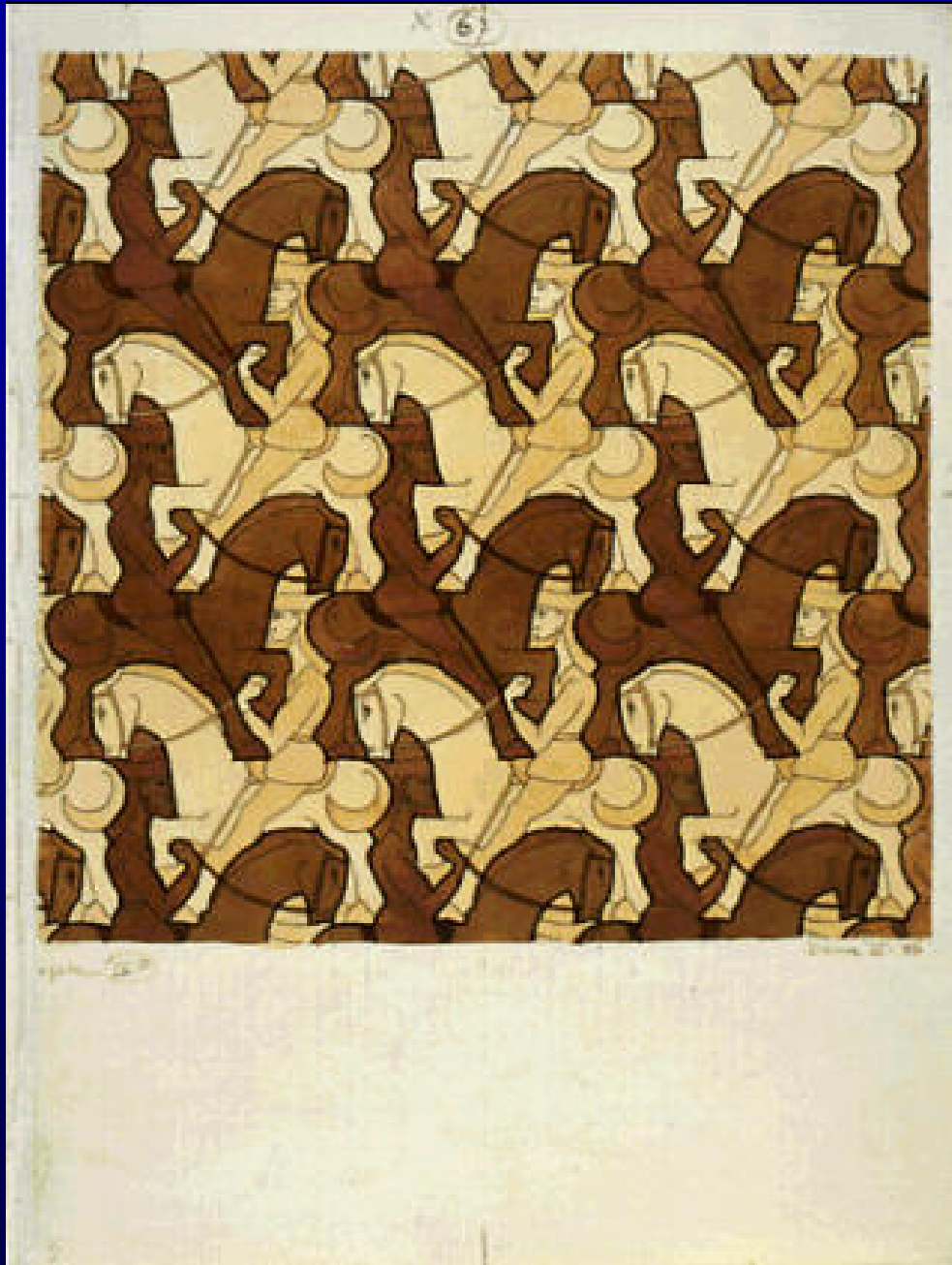
Symmetry:

spatial (taiji, Eight trigram..), temporal,
matter-antimatter, gauge
dynamic

Invariance of physics laws under some
symmetry operation

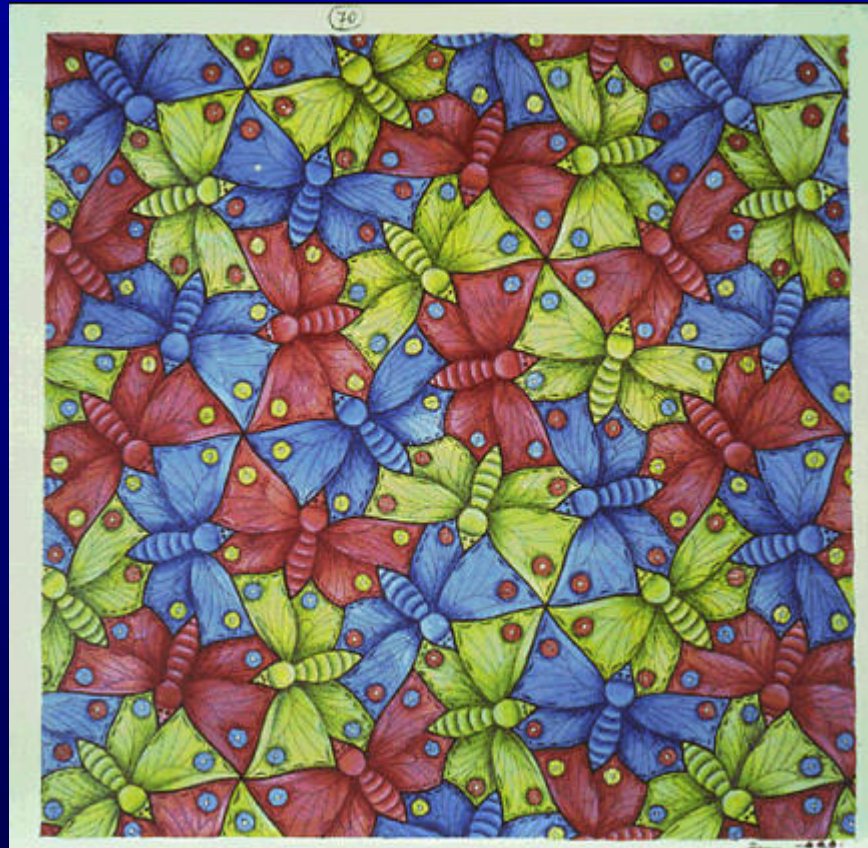


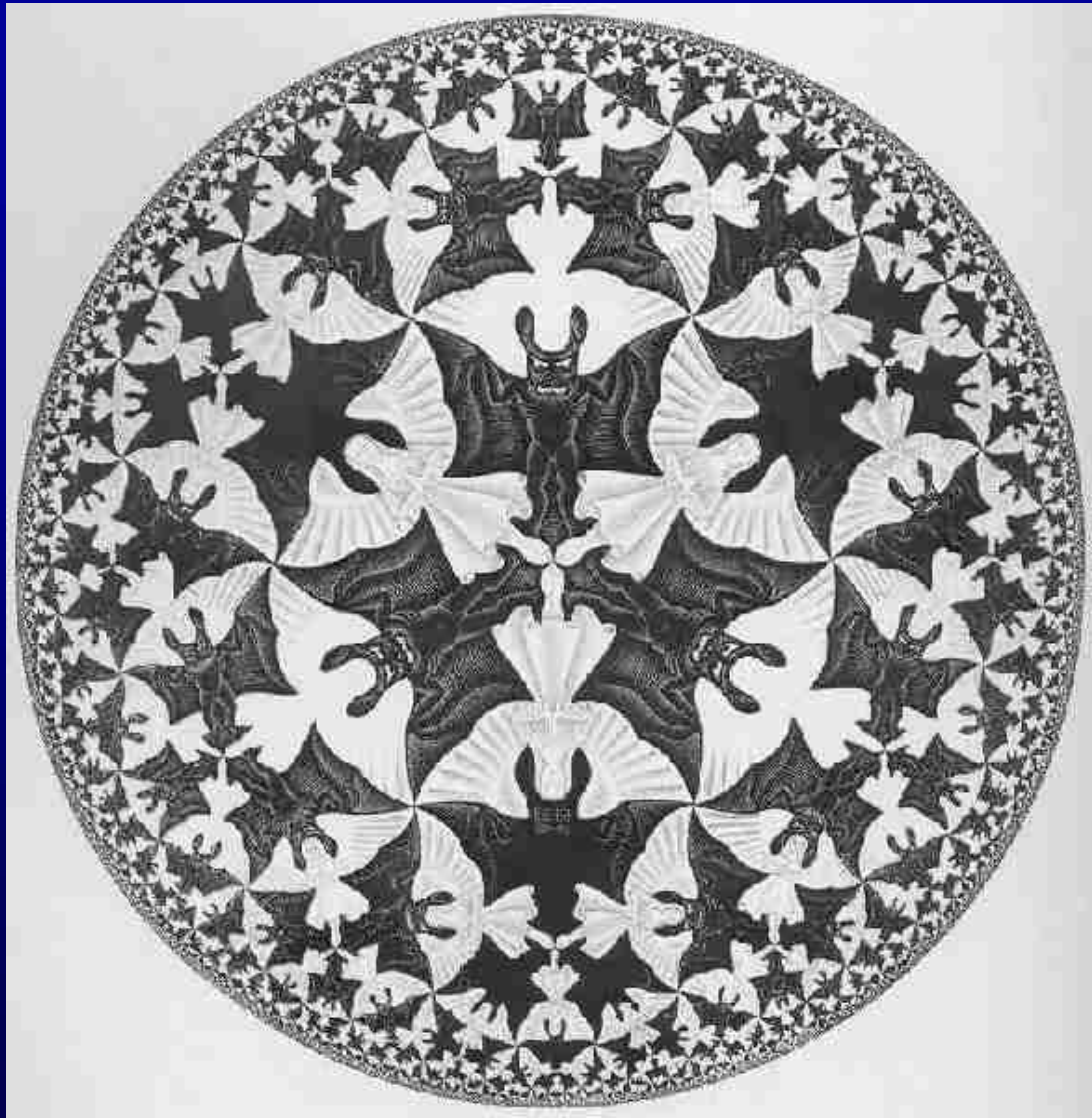
Day and Night by Escher M C (1898-1972)

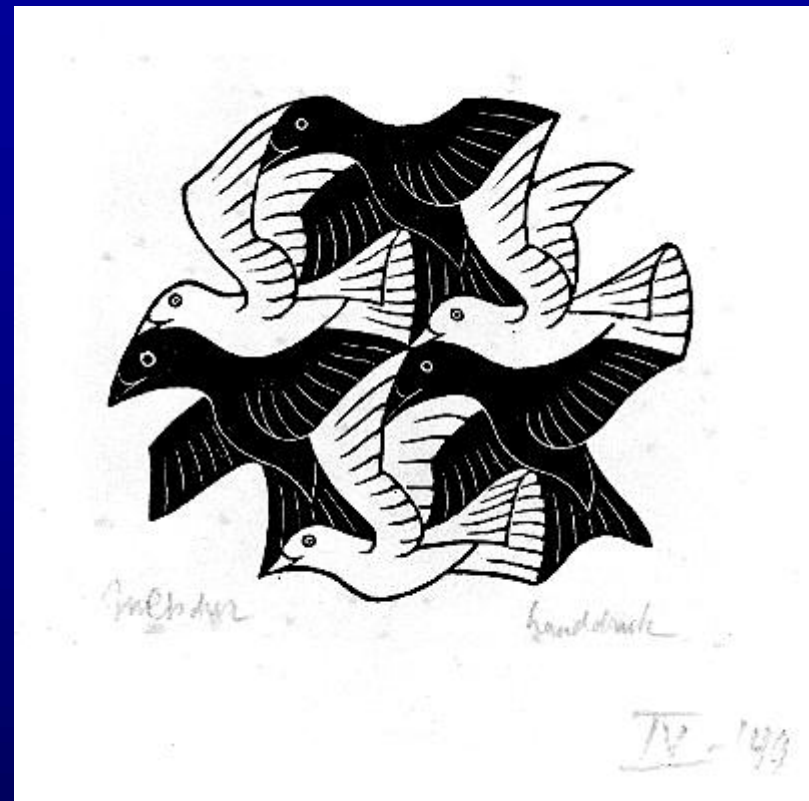


Optical Illusions

Geometric transformation:
translation, reflection, rotation etc.







<http://www.mindspring.com/~mc.escher/escher.html>





fern (蕨)

self-similarity

fractal

Simplicity

fundamental laws of nature

Okemah's razor

(William of Ockham, 14th century

English logician)

**Of two competing theories or
explanations, all other things being equal,
the simpler one is to be preferred.**

(Entities should not be multiplied unnecessarily)

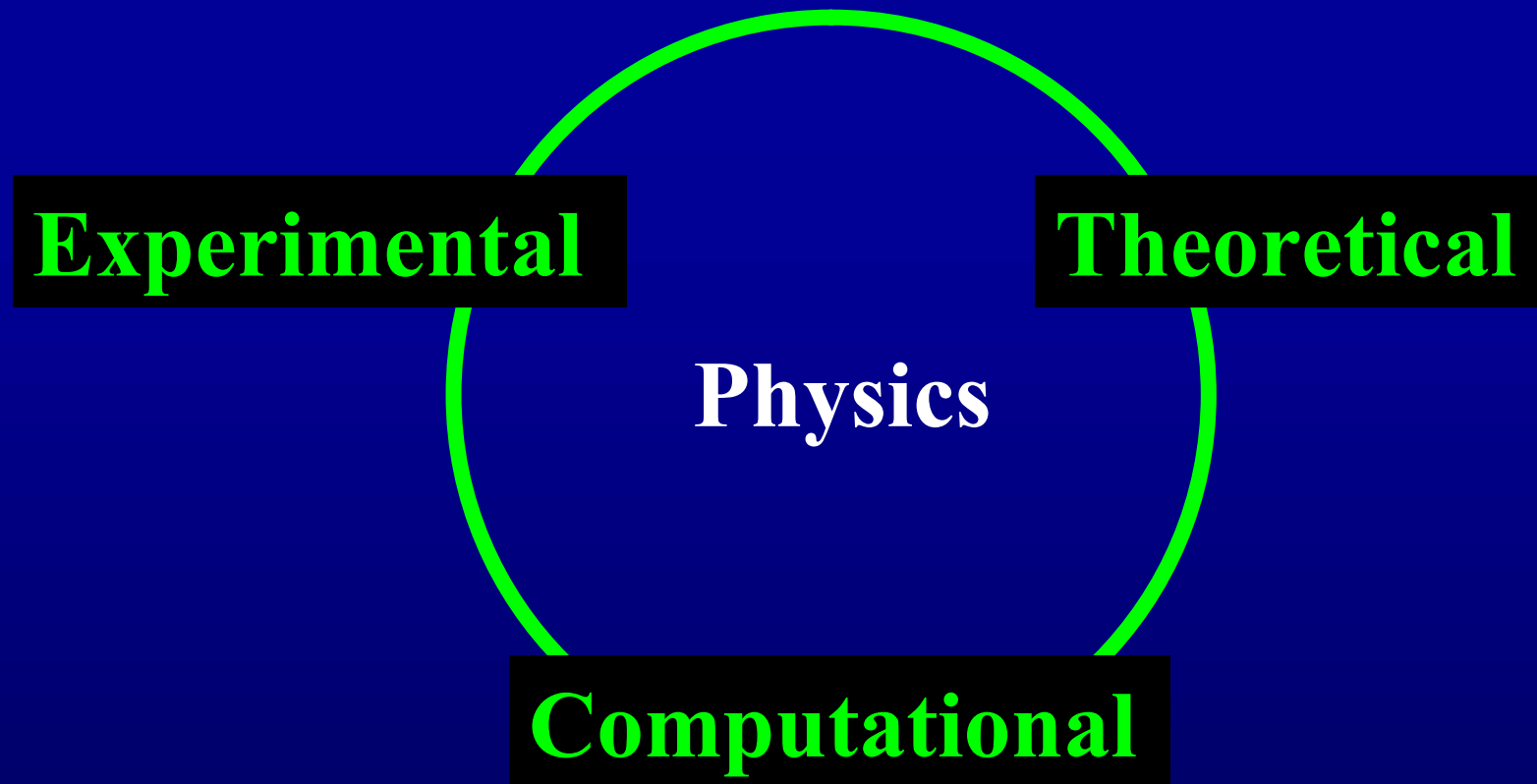
Matter:

Field propagates interaction.

It is a special form of matter. This is even more distinct in advanced theory:

EM Interaction → Exchange of photon

Strong Interaction → Exchange of gluon



Experiments provide
enough variation and flexibility
e.g. temperature up to 4.2K, nK!
Molecular assembly

Five Great Theories in physics:

- **Newtonian Mechanics (classical mechanics)**
- **Thermodynamics**
- **Electromagnetism**
- **Relativity**
- **Quantum mechanics**

**None will ever be completely overthrown.
None will prove to be entirely correct. No
theory is unique...**

- Physics is the most **fundamental** science.
- Physics is the most developed science.
- Physics provides most and fundamental means of scientific research.

Reductionism 还原论

Normal scientific method:

hypothesis
pl. hypotheses

1. Observation

2. Formulation of hypothesis (假说)

3. Predict other phenomena or quantitative results

4. Perform experimental tests to verify/nullify the hypothesis (证实 / 证伪) (falsify)

Karl Raimund Popper(1902 – 1994)

A theory should be accounted scientific if and only if it is falsifiable.

Conjectures and Refutations

Abnormal?

Intuition(premonition)

Dumb luck (serendipity)

To ignore alleged facts:

not really fact;

irrelevant or inconsistent;

mask other more important facts;

complicating a situation.

Einstein? Michelson-Morley experiment

At college level you must do **more than learn facts, laws, equations and problem-solving techniques.**

- to seek to grasp the whole of physics,**
- to appreciate its generality,**
- to get an idea of the beauty, simplicity, harmony, and grandeur of some of the basic physical laws,**
- to see the interconnection of its parts and perceive its boundaries.**

- to distinguish between
theory and application,
physical idea and mathematical tools,
general law and specific fact,
dominant and irrelevant effect,
traditional and modern reasoning.

*"The most incomprehensible thing
about the world is that it is comprehensible."*

Albert Einstein

1.2 Physical quantities

A general theory of physics is made up of

- a set of **concepts**, usually not directly perceptible
- assumption about the mathematical representation (**physical quantities**)
- relationships among physical quantities (**equation**)

A certain mathematical representation of a physical quantity is never natural and assumptions are necessary.

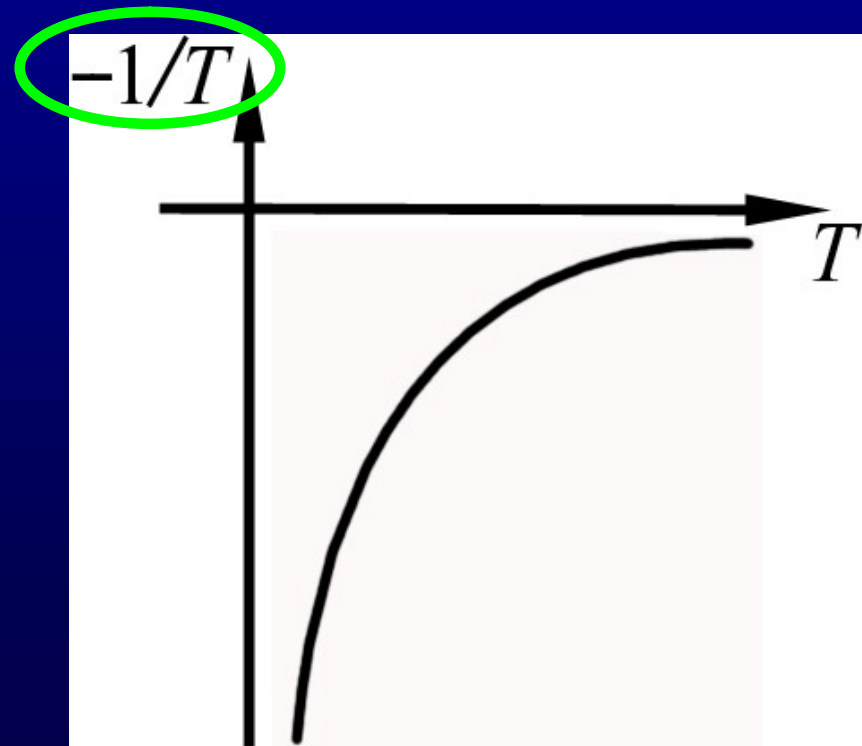
Personal preference or historical accident

e.g., Temperature:

Celsius(0°C),

Fahrenheit,

Kelvin



Anders Celsius, 1701-1744, Swedish astronomer.

He firstly recommended in 1742 to divide the temperature scale of a mercury thermometer at 760mm mercury air pressure into 100 degrees, where 100 is the freezing point and 0 the boiling point of water . Later the reversion of the Celsius scale with 0 as the frozen point and 100 as the boiling point of water was introduced .

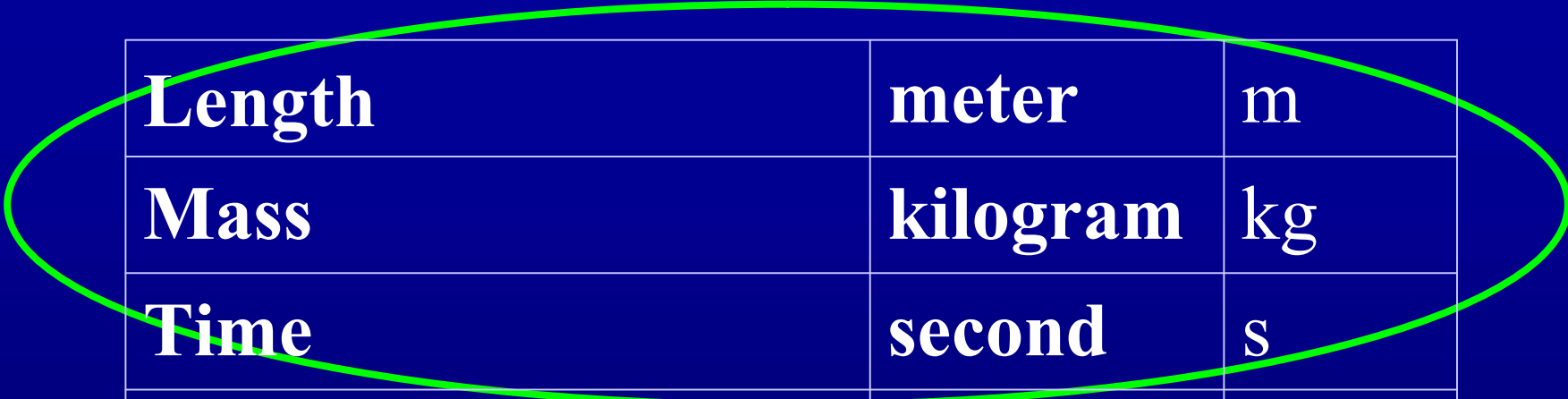
Basic quantities are defined through
measurement (operational definition)
standard & unit, procedure

Derived quantities are defined in terms of
other quantities $1\text{N} = 1\text{kg}\cdot\text{m}/\text{s}^2$

Basic or Derived — Arbitrary!

Electric quantity Q or current I basic?

SI base unit



Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Le **S**ystème **I**nternational d'Unités

Length:

1889 standard meter (platinum iridium meter bar)

1960 atomic standard ^{86}Kr

1983 $c = 299\,792\,458\text{ m/s}$

BIPM (Bureau International des Poids et Mesures)
国际度量衡局 <http://www.bipm.org/>

Typical lengths: table 1-2 10^{61}

Moving body?

Space quantized—fundamental length

一尺之杵，日取其半，万世无穷



A replica of the platinum-iridium bar in Paris that was the international standard for length before 1960 .

Non-SI units

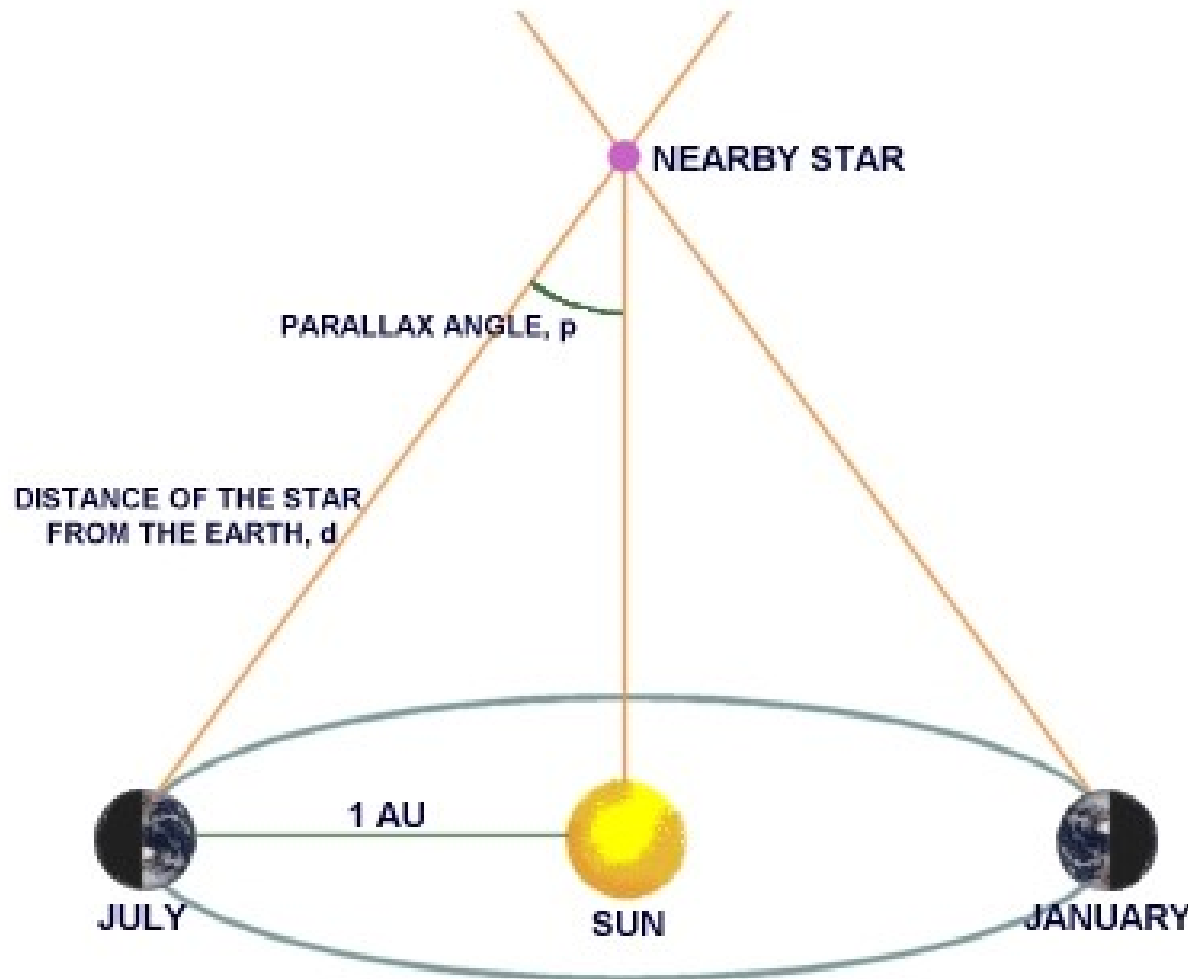
AU(Astronomical unit) $1.495\,978\,70 \times 10^{11}\text{m}$

light year $9.460\,530 \times 10^{15}\text{m}$

pc, parsec, parallax second $3.261\,633\text{ l.y.}$

Å, Ångström $1.0 \times 10^{-10}\text{m}$

Swedish physicist Anders Jonas Ångström
(1814–1874)



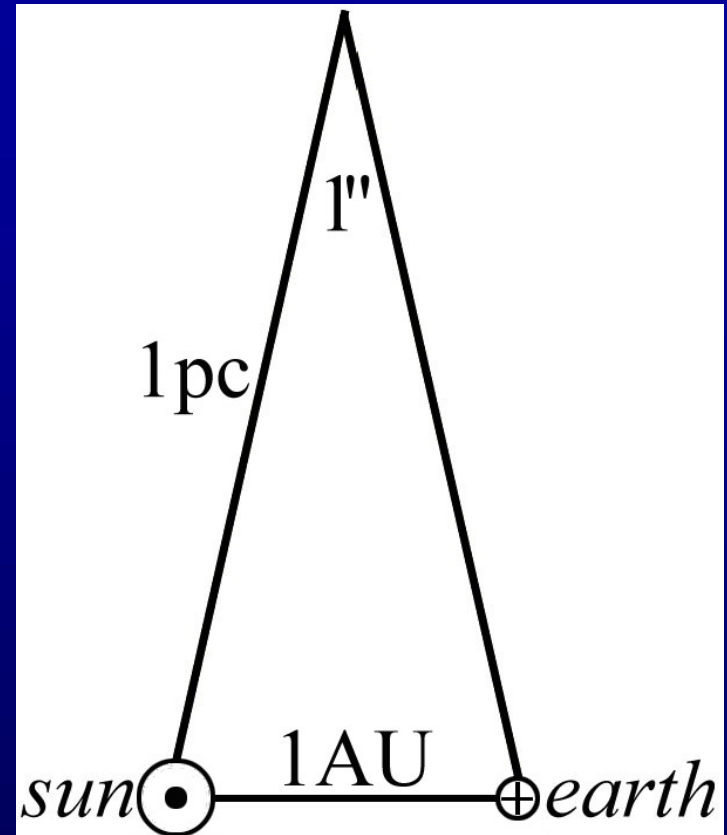
The parallax can be used to measure the distance to the few stars which are close enough to the Sun to show a measurable parallax. 视差法

Example 1.1

$$1\text{pc} = \frac{1\text{AU}}{1'' \frac{1^\circ}{3600''} \frac{\pi}{180^\circ}}$$

$$= 206265\text{AU}$$

$$= 3.086 \times 10^{16} \text{m} = 3.262\text{ly}.$$



Mass

Standard kilogram

unified atomic mass unit

$$1 \text{ u} = \frac{1}{12} m_{^{12}\text{C}}$$

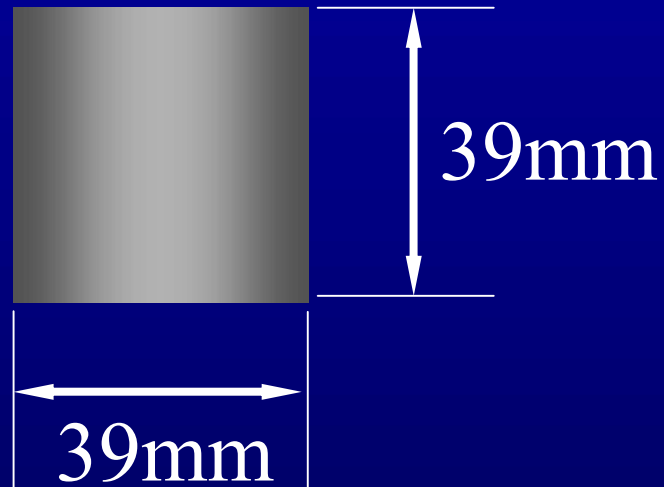
$$1 \text{ u} = 1.660538\,73(13) \times 10^{-27} \text{ kg}$$



The international prototype of kilogram (IPK) made of platinum-iridium, is kept at the BIPM under conditions specified by the 1st CGPM in 1889. CGPM---Conference Generale des Poids et Mesures (General Conference on Weights and Measures)
国际度量衡大会

kg No.60 90% Pt + 10% Ir

(No. 64)



1965: 1 kg + 0.271 mg

1989 ~ 1993: 1 kg + 0.295 mg

“last unit to dematerialize”

“We get rid of the last artifact.”

IPK $\longrightarrow$ “electronic kilogram”

Kibble balance (Bryan Kibble)

https://en.wikipedia.org/wiki/Kibble_balance

fixing the Planck constant h be equal to exactly
 $6.626070040 \times 10^{-34}$

It is expected that the definition of the kilogram and several other units will change on May 20, 2019, following a final vote by the CGPM in November 2018.

Proposed redefinition of SI base units

http://en.wikipedia.org/wiki/Proposed_redefinition_of_SI_base_units

kilogram:

fixing the Planck constant h be equal to exactly
 $6.626070040 \times 10^{-34}$

mole :

fixing the Avogadro constant N_A to be equal to exactly
 $6.022140857 \times 10^{23} \text{mol}^{-1}$

ampere:

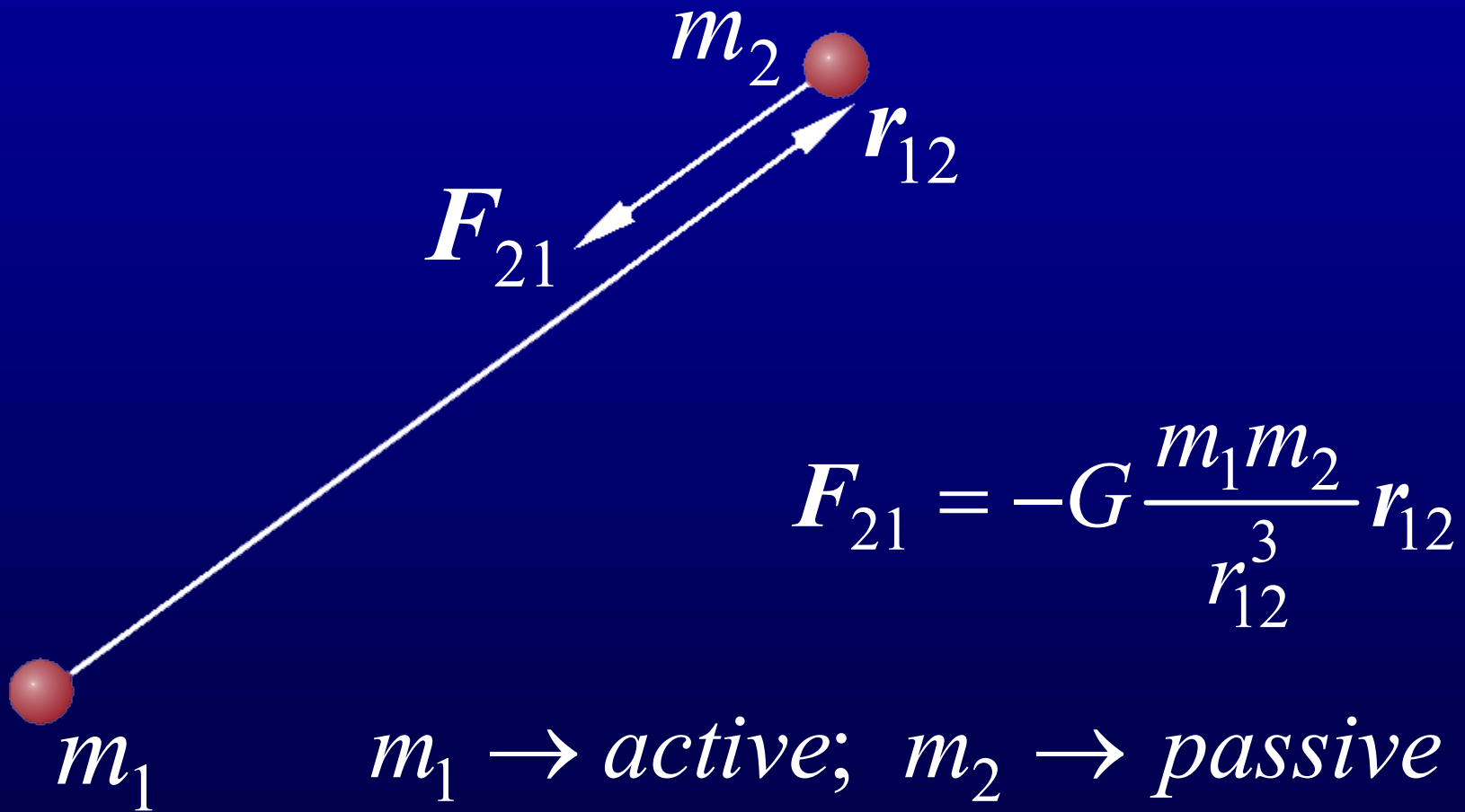
fixing the elementary charge e to be equal to exactly
 $1.6021766208 \times 10^{-19} \text{ C}$

kelvin:

fixing the Boltzmann constant k to be equal to exactly
 $1.38064852 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$

Gravitational mass vs inertial mass

active, passive



Mass is motion-dependent: $m = m(v)$

Rest mass $m_0 = m(0)$

Does rest mass conserve?

Fission and fusion

Particles moving with c $m_0 = 0$

$m_\nu = 0?$

neutrino

中微子

Time IBWM (International Bureau of Weight and Measures)

1second = (1/86400) mean solar day

1956 IBWM:

1/31556925.9747 of

a standard tropical year—1900

Atomic standard ^{133}Cs (since 1967)

回归年

A clock counts repeated events of motion.

A moving clock runs slower.

Twin paradox

Unidirectivity of time, arrow,

佯谬

Universal physical constants:

$$h = 6.626\,069\,57(29) \times 10^{-34} \text{ J} \cdot \text{s}$$

reduced planck
constant

$$\hbar = \frac{h}{2\pi} = 1.054\,571\,726(47) \times 10^{-34} \text{ J} \cdot \text{s}$$

$$G = 6.67384(80) \times 10^{-11} \text{ N} \cdot \text{m}/\text{kg}^2$$

uncertainty

Gamow G, Stannard R. *The New World of Mr. Tompkins*. Cambridge University Press 1999.

中译本：伽莫夫，斯坦纳德．物理世界奇遇记(最新版)．吴伯泽译．长沙：湖南教育出版社，2000

Dimension and dimensional analysis

量纲、维度

Geometrically **area** $\sim (\text{length})^2$

volume $\sim (\text{length})^3$

physical quantity $\sim (\text{basic quantity})^d$

Base dimensions

dimension
symbols

Length	L
Mass	M
Time	T
Electric current	I
Temperature	θ
Amount of substance	N
Luminous intensity	J

Dimension and dimensional analysis

Geometrically **area** $\sim (\text{length})^2$

volume $\sim (\text{length})^3$

physical quantity $\sim (\text{basic quantity})^d$

量纲、维度

$$\text{mechanical quantity} \sim L^{\alpha} M^{\beta} T^{\gamma}$$

e.g. $[\text{velocity}] = L^1 T^{-1}$ or $\text{dim}(\text{velocity})$
 $[\text{acceleration}] = L^1 T^{-2}$
 $[\text{force}] = M L^1 T^{-2}$

Hubble constant $H_0 = 50 \sim 100 \text{ km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$
 $\text{dim } H_0^{-1} = T$

Dimensionally consistent

Dimensionless quantity:

$$\text{Dim} () = L^0 M^0 T^0 = 1$$

$$\beta = \frac{1}{k_B T}$$

Reynolds number $Re = \rho v d / \eta$

Plasma parameter $\Gamma = \beta e^2 / \bar{a}$

Reynolds number laminar flow 层流

turbulent flow 湍流

The slowdown rate $\sim$ s/day, s/century

dimension $\longleftrightarrow$? $\longleftrightarrow$ **unit**

Example 1.2

Use constants $G, \hbar$ and c to construct a quantity with the dimension of the length.

Solution: Set $l_g = G^\alpha \hbar^\beta c^\gamma$, then

$$\dim G^\alpha \hbar^\beta c^\gamma = L^1.$$

Noting that

$$\dim c = L^1 T^{-1}$$

$$\dim \hbar = M L^2 T^{-1}$$

$$\dim G = \dim Fr^2 / m^2 = M^{-1} L^3 T^{-2}$$

Example 1.2

we have

$$L^{3\alpha+2\beta+\gamma} M^{-\alpha+\beta} T^{-2\alpha-\beta-\gamma} = L^1.$$

By comparing the exponents of two sides, we have

$$\alpha = \beta = \frac{1}{2}, \quad \gamma = -\frac{3}{2}.$$

Therefore

$$l_g = \left(\frac{G\hbar}{c^3} \right)^{1/2} \approx 1.6 \times 10^{-35} \text{ m.}$$

Planck length

Example

Known gravitational system of m, R

Find a quantity of dimension energy

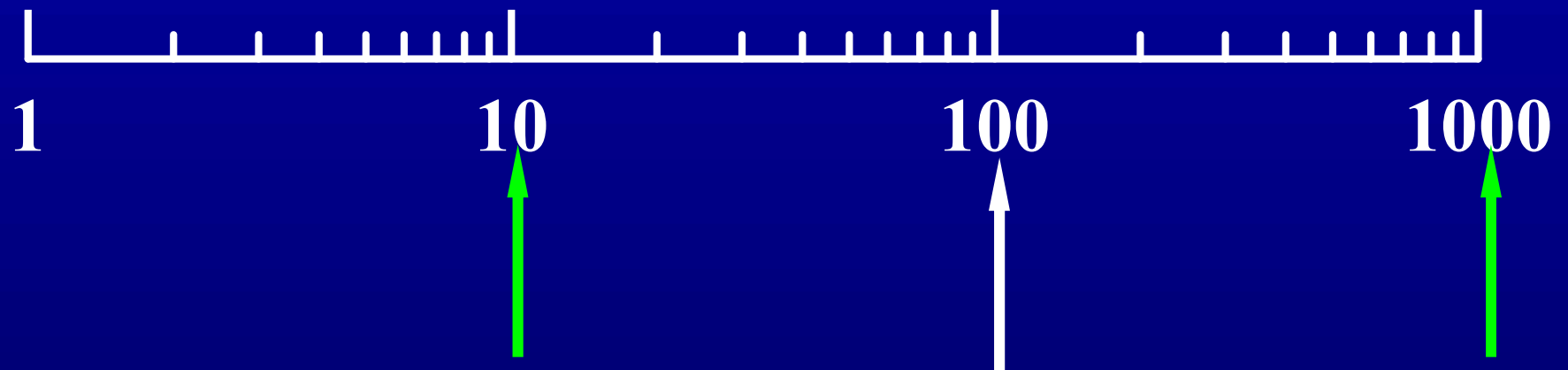
$$U \sim W \sim F \cdot r \sim -G \frac{m_1 m_2}{r^2} \cdot r$$

$$U = -G \frac{m^2}{R}$$

The gravitational self-energy

1. **spherical: single parameter R means spherical,**
2. **G is introduced due to “gravitational” ,**
3. **either negative sign or positive sign is OK.**
4. **meaning of the quantity: broken to sheet.**
5. **numerical factor is model-dependent, beyond the dimensional analysis: uniform sphere 0.6, thin shell 0.5**
6. **physical systems are divided into layers, certain layer is referred.**

Problem 1.5 ‘logarithmic scale’



linear scale

1.3 Approximation in physics

Correct and ingenious use of approximation is fundamental for scientific workers.

Exact solution $\longleftrightarrow$ *model approximation*
numerical computation, math approximation

$g = \text{constant?}$

$$F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2}$$

Math approximation $finite \rightarrow \begin{cases} \text{not infinite} \\ \text{not null} \end{cases}$

If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = C, \quad 0 < |C| < \infty$

f, g are of same infinitesimal **order**

$$\longrightarrow f(x) = O(g(x))$$

$$3x^2 + 5x^4 = \begin{cases} O(x^2) & x \rightarrow 0 \\ O(x^4) & x \rightarrow \infty \end{cases}$$

If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$

f is infinitesimal relative to g

much smaller than


$$f(x) = o(g(x))$$

Usually we say $a \ll b$ ($a \gg b$) if a is **two** order smaller (greater) than b

$$e^{-1} \ll 1 \quad \text{means ...}$$

Taylor series:

$$f(x_0 + x) = f(x_0) + f'(x_0)x + \frac{1}{2!}f''(x_0)x^2 + \dots$$

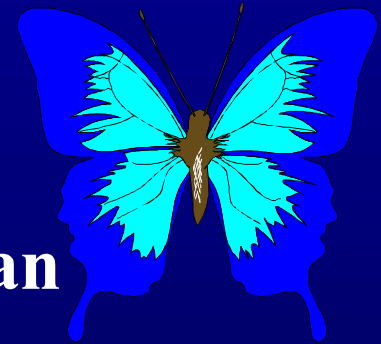
$$\frac{1}{1-x} = 1 + x + x^2 + \dots \quad x < 1$$

$$(1+x)^q = 1 + qx + \frac{1}{2!}q(q-1)x^2 + \dots \quad x < 1$$

For nonlinear system, problem is sensitive to initial values.

**“Butterfly effect”:
chaos theory**

**Edward Lorenz , American mathematician
and meteorologist**



**The flap of a butterfly's wings in Brazil might
set off a tornado in Texas .**

1.4 Vectors

矢量、向量

Scalars: magnitude (a number & a unit) e.g.,
 $m, l, t, \rho, E, T...$ 标量

Vectors have magnitude, direction, &
combine according to specific rules

e.g., $\Delta r, v, a, \omega, M, E...$

Hand-writing

$\Delta \vec{r}, \vec{v}, \vec{a}, \vec{\omega}, \vec{M}, \vec{E}...$



an old-fashioned way to denote a vector
--a tilde underneath a letter

$$\mathbf{e}_A = \mathbf{A} / A \quad \text{Unit vector} \quad A = |\mathbf{A}|$$

$$\mathbf{r} = \mathbf{r}(t), \mathbf{W} = \mathbf{W}(\mathbf{F}, \Delta \mathbf{r})$$

position vector

displacement vector

Representations of vector

- geometric: directed line segment
- analytic e.g.

$$\mathbf{A} = (A_1, A_2, A_3)$$

- *tensor of rank one

一阶张量

解析的

Combination rules

1. Addition :

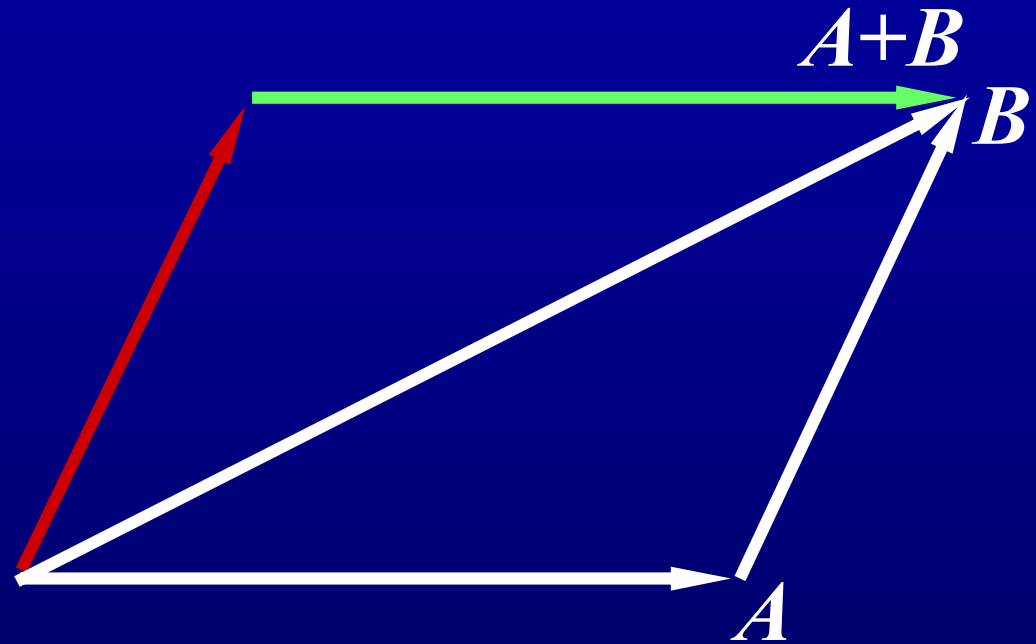
parallelogram law or triangle law

Combination rules

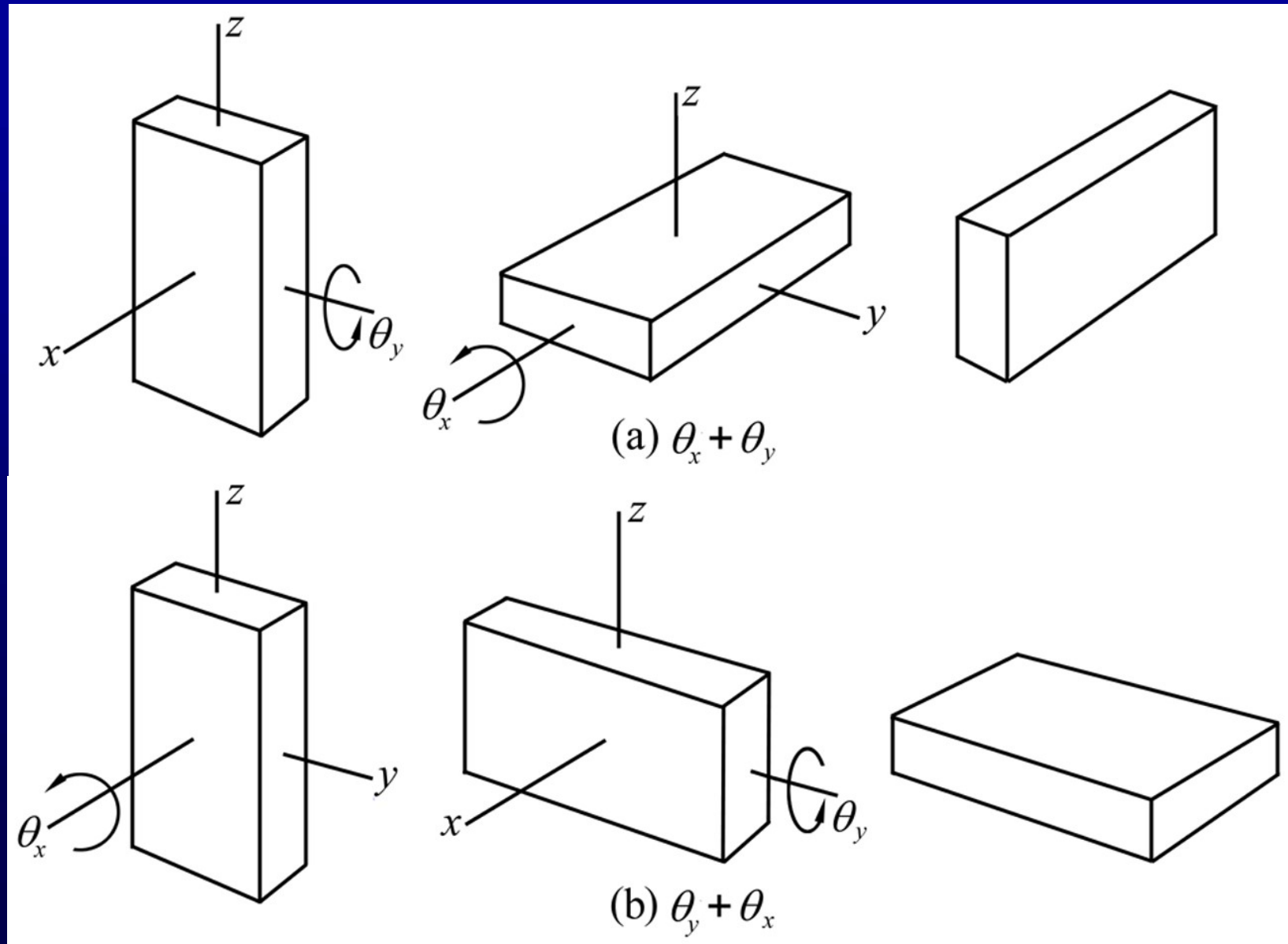
1. Addition :

- Commutative

$$A + B = B + A$$



$$\theta_x + \theta_y = \theta_y + \theta_x ?$$



Combination rules

1. Addition:

- Commutative

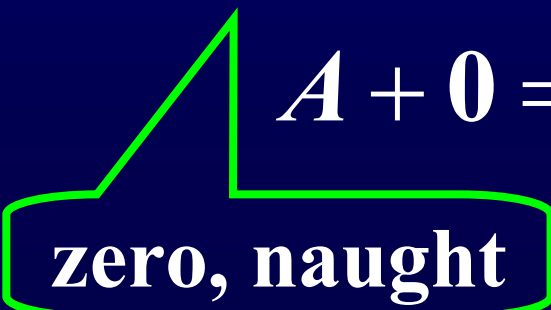
$$A + B = B + A$$

- associative

$$A + (B + C) = (A + B) + C$$

- null vector

$$A + \mathbf{0} = A$$



zero, naught

2. Multiplication by a scalar

$$\lambda A = C; \quad C = |\lambda|A$$

$$\lambda > 0 \quad \text{parallel to } A$$

$$\lambda < 0 \quad \text{anti-parallel to } A$$

- **associative** $\lambda(\mu A) = (\lambda\mu)A$

- **distributive** $(\lambda + \mu)A = \lambda A + \mu A$

$$\lambda(A + B) = \lambda A + \lambda B$$

$$A - B = A + (-1 \times B)$$

3. Scalar (dot, inner) product

$$A \cdot B = AB \cos \theta \quad (\text{a scalar function})$$

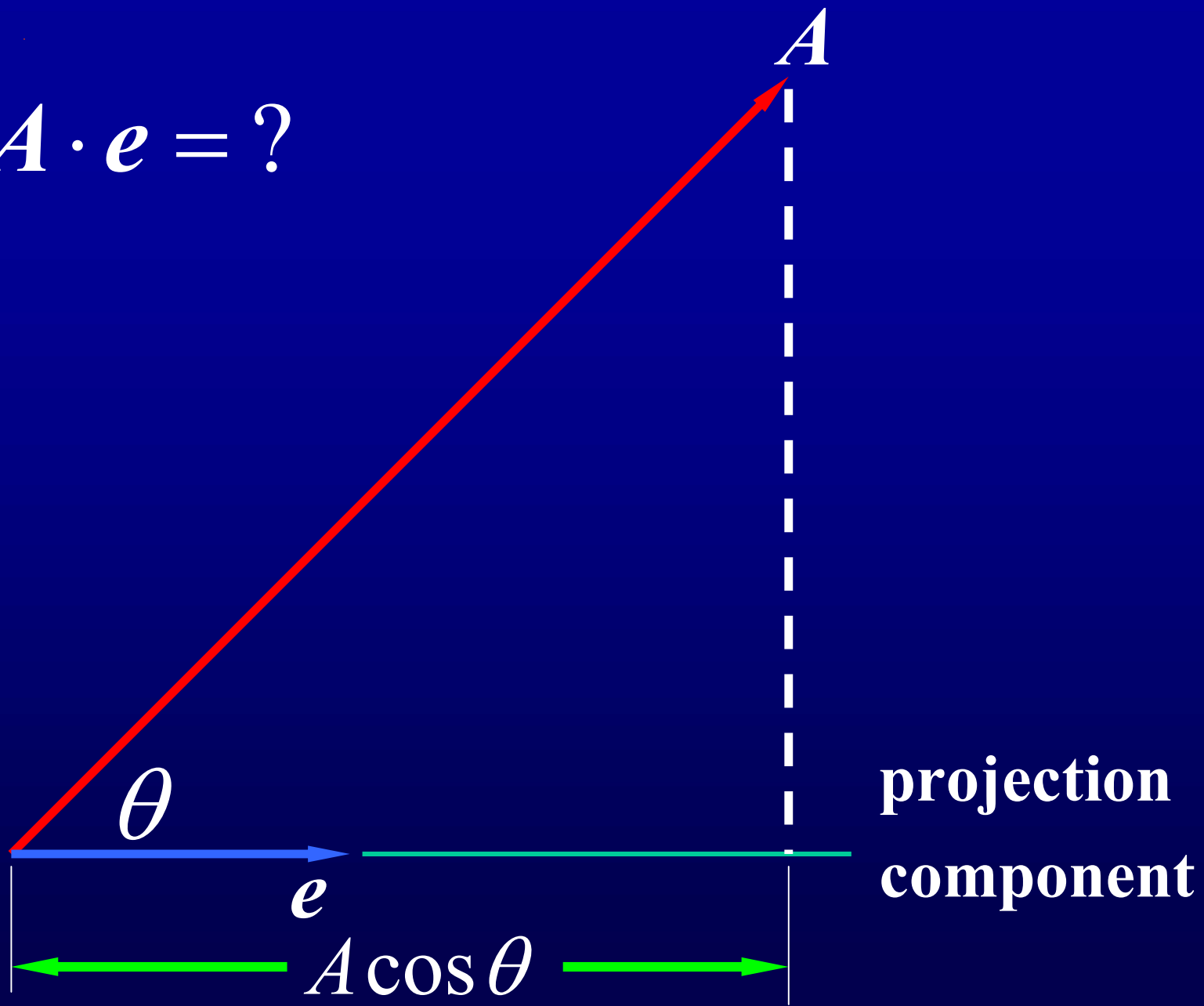
$$A \cdot B = B \cdot A$$

$$A \cdot (\alpha B + \beta C) = \alpha A \cdot B + \beta A \cdot C$$

$$A \cdot A = A^2 \geq 0$$

What does $A \cdot B = 0$ mean?

$$A \cdot e = ?$$



Resolution

$$A = A_1 e_1 + A_2 e_2$$

A, e_1, e_2 coplanar e_1, e_2 not collinear

$$e_1 \cdot A = A_1 (e_1 \cdot e_1) + A_2 (e_1 \cdot e_2)$$

$$= A_1 (e_1 \cdot e_1) \quad \text{when } e_1 \cdot e_2 = 0$$

$$= A_1 \quad \text{when } e_1 \cdot e_1 = 1$$

$$e_2 \cdot A = A_1 (e_2 \cdot e_1) + A_2 (e_2 \cdot e_2)$$

$$= A_2 \quad \text{when } e_2 \cdot e_2 = 1$$

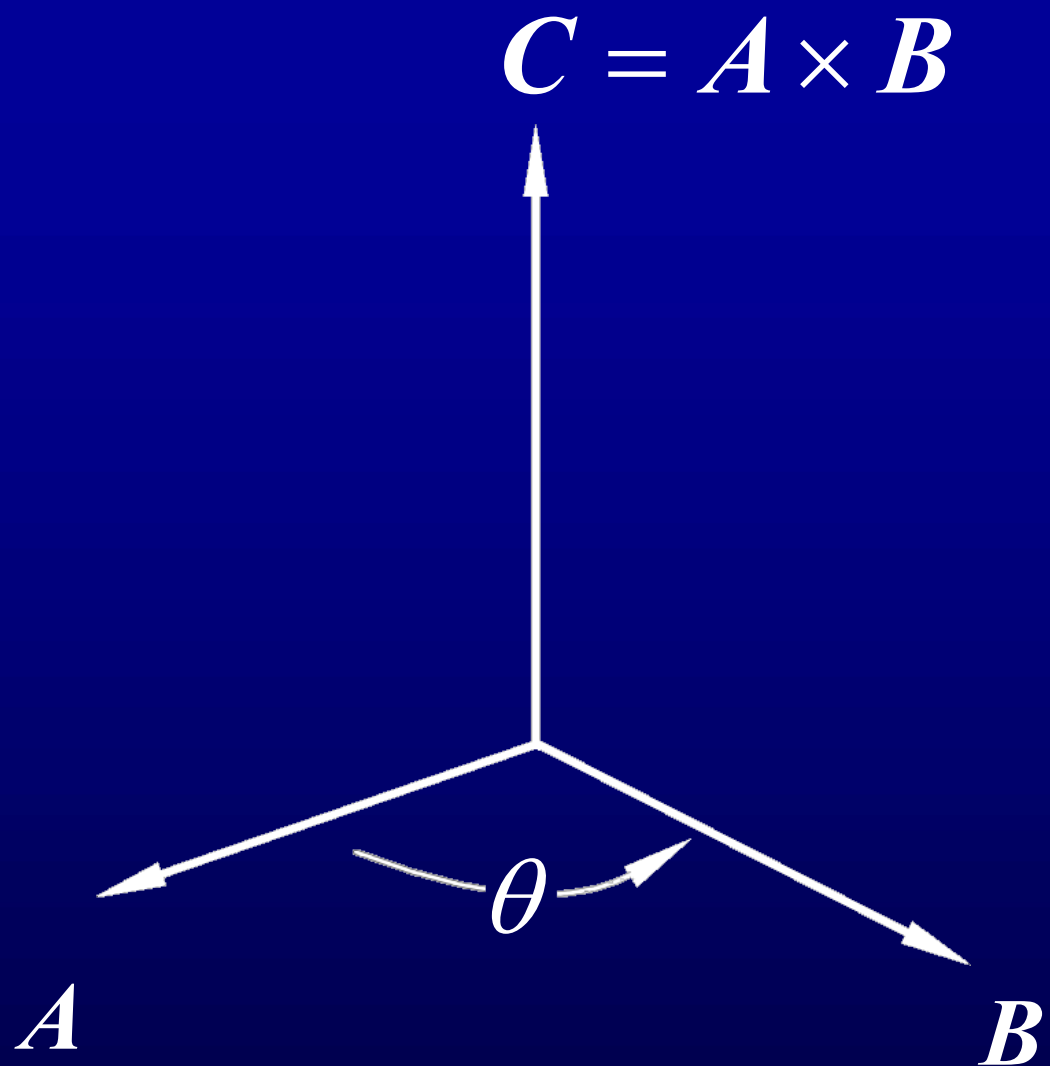
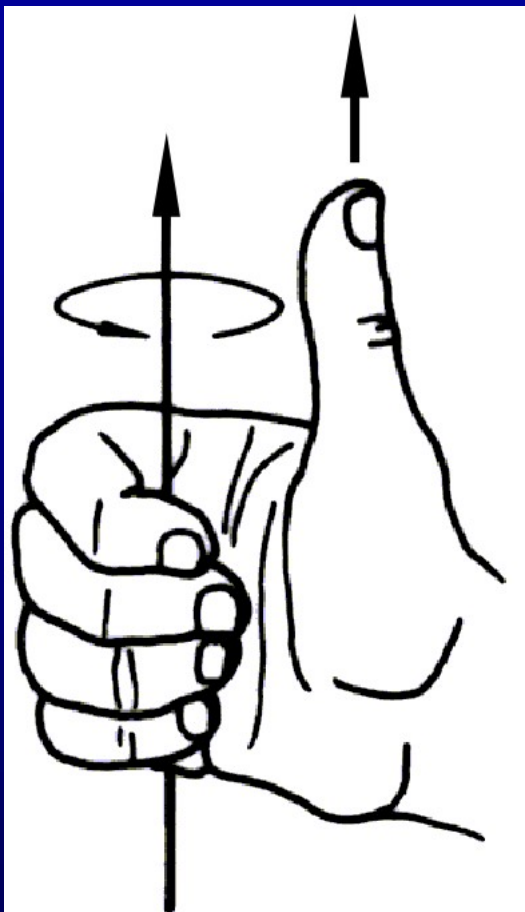
***In three-dimensional space?**

4. Vector (cross, external) product

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta \quad (0 \leq \theta \leq \pi)$$

$\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in right-handed screw



4. Vector (cross, external) product

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta \quad (0 \leq \theta \leq \pi)$$

$\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in right-handed screw

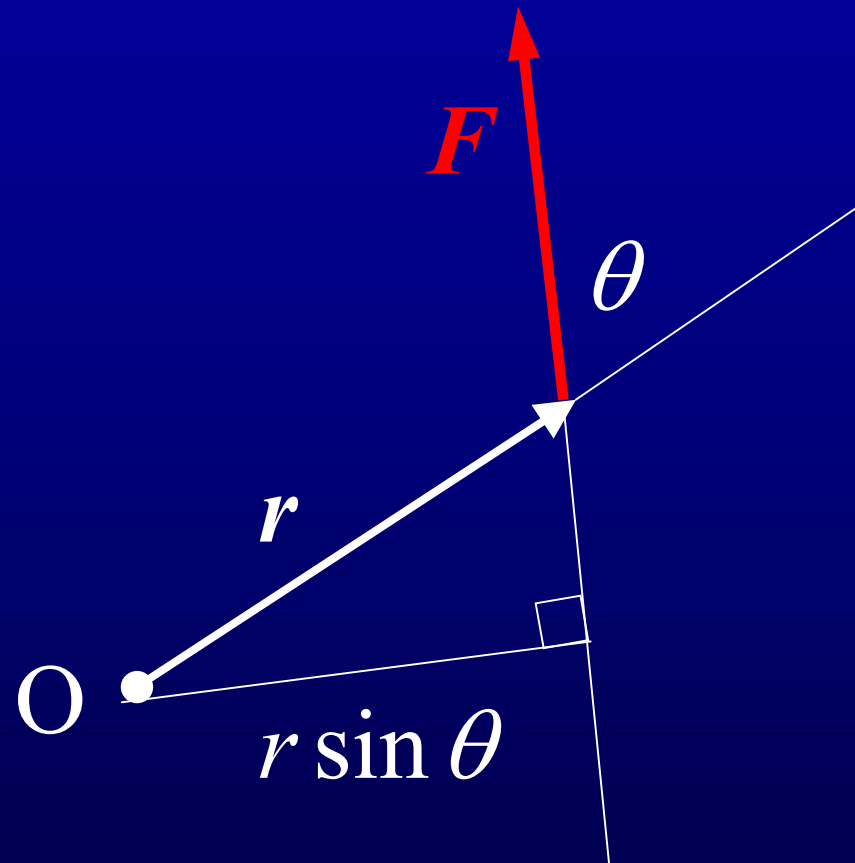
$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times (\alpha \mathbf{B} + \beta \mathbf{C}) = \alpha \mathbf{A} \times \mathbf{B} + \beta \mathbf{A} \times \mathbf{C}$$

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

Motivations for the Cross Product

Torque

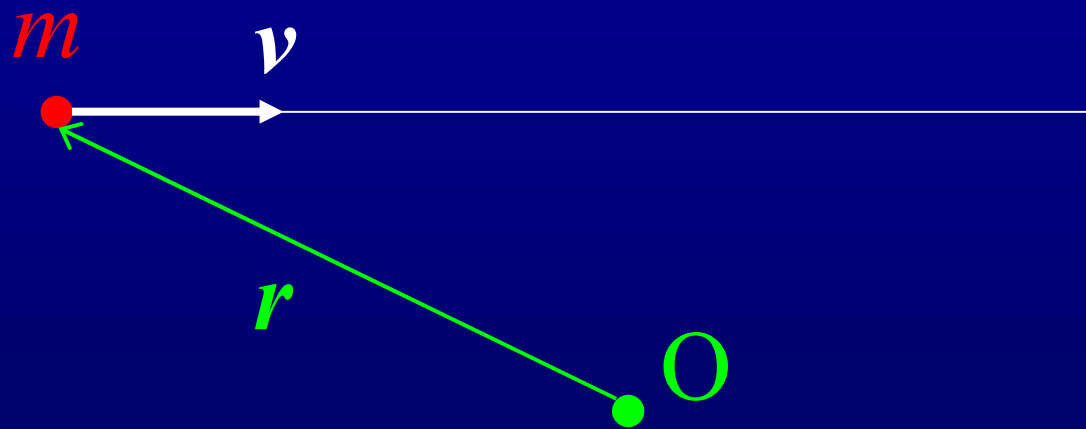


$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

$$M = Fr \sin \theta$$

Motivations for the Cross Product

Angular momentum



(Linear) momentum: $p = mv$

Angular momentum: $L = r \times p$

space inversion $r \leftrightarrow -r$

$$F \leftrightarrow -F$$

polar vector

$$p \leftrightarrow -p$$

polar vector

$$M \leftrightarrow M$$

axial vector

$$L \leftrightarrow L$$

axial vector

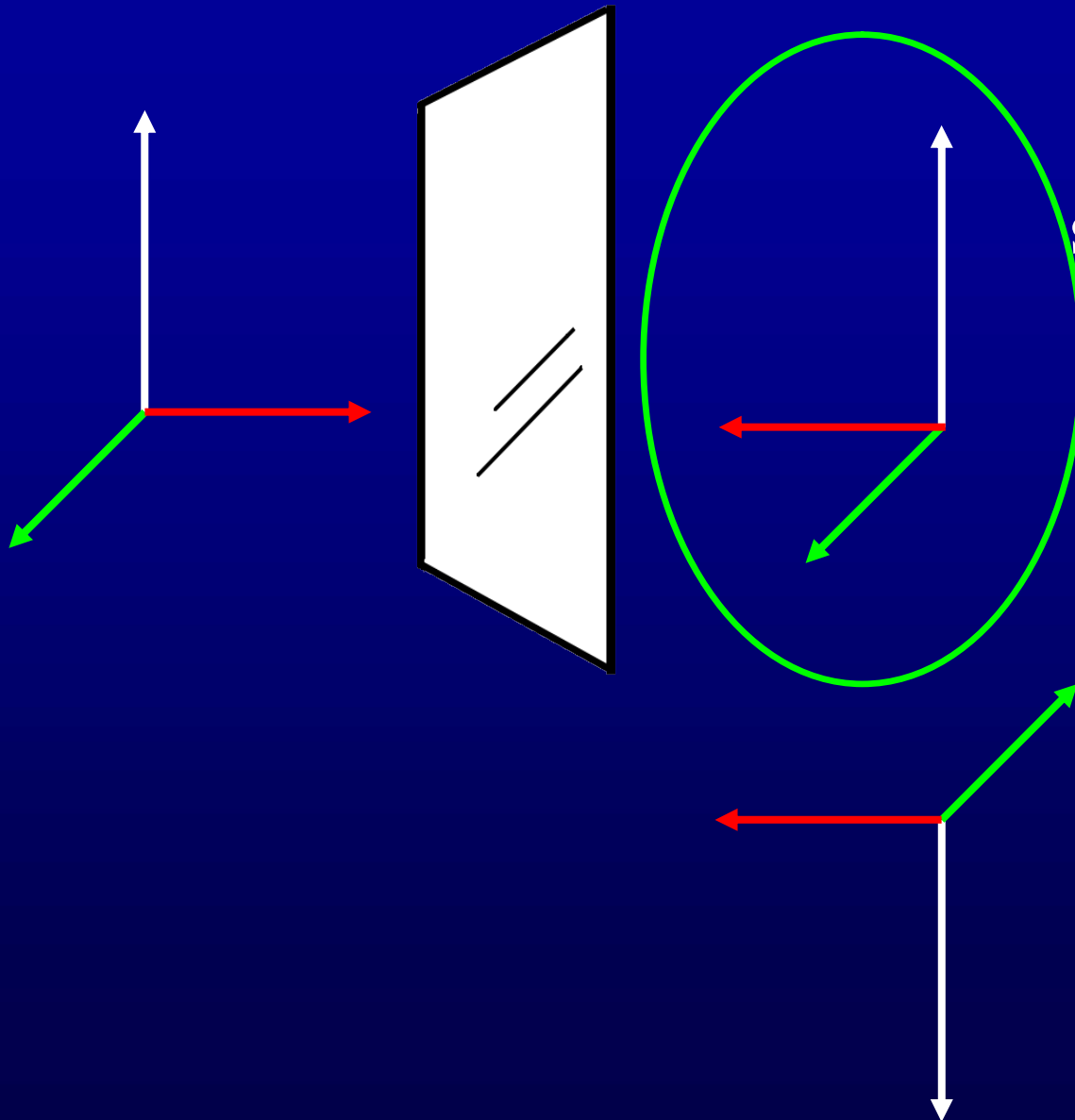
pseudo vector

$$M = r \times F$$

$$L = r \times p$$

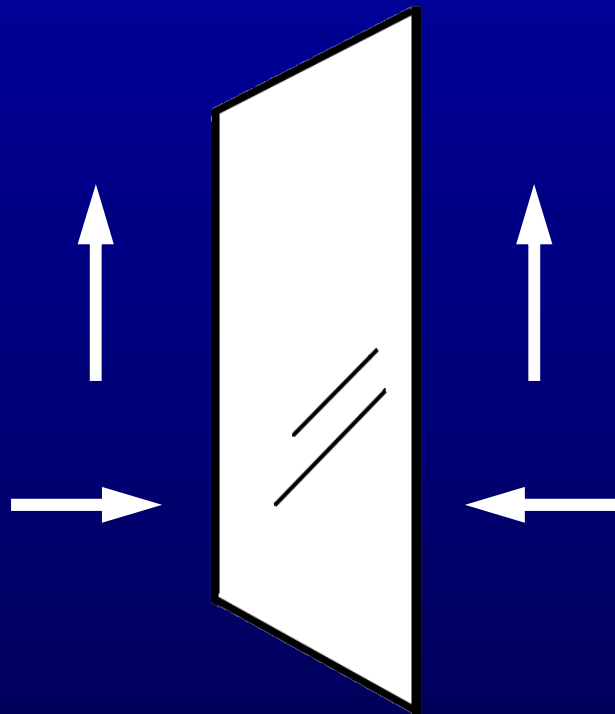
space inversion

$$\mathbf{r} \leftrightarrow -\mathbf{r}$$

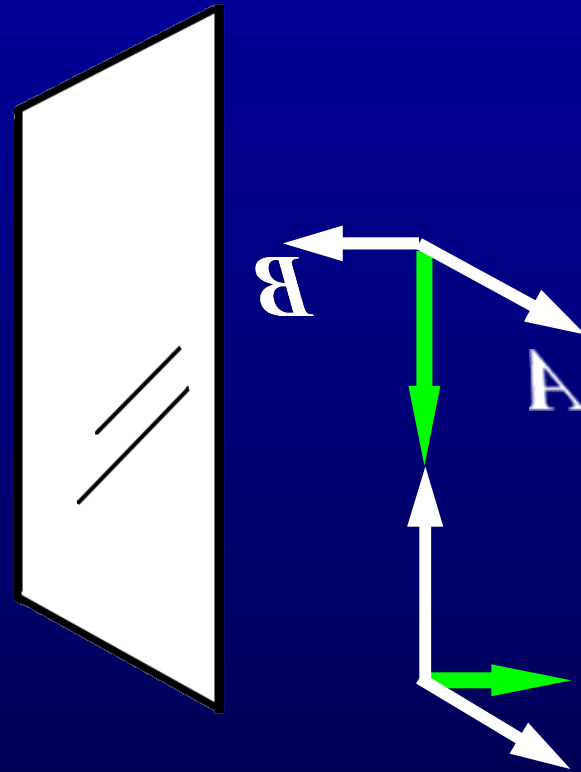
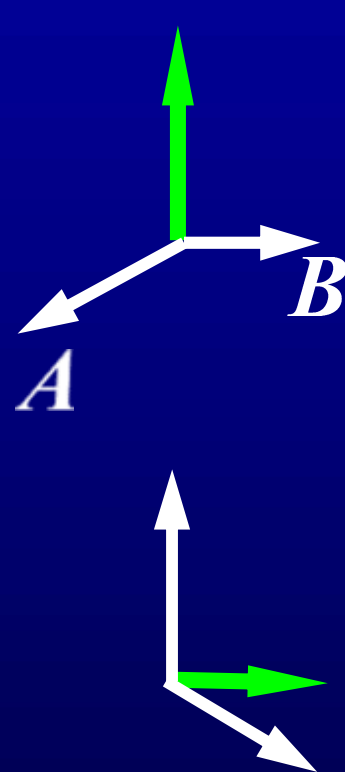


space inversion
=
mirror image
+
180° rotation

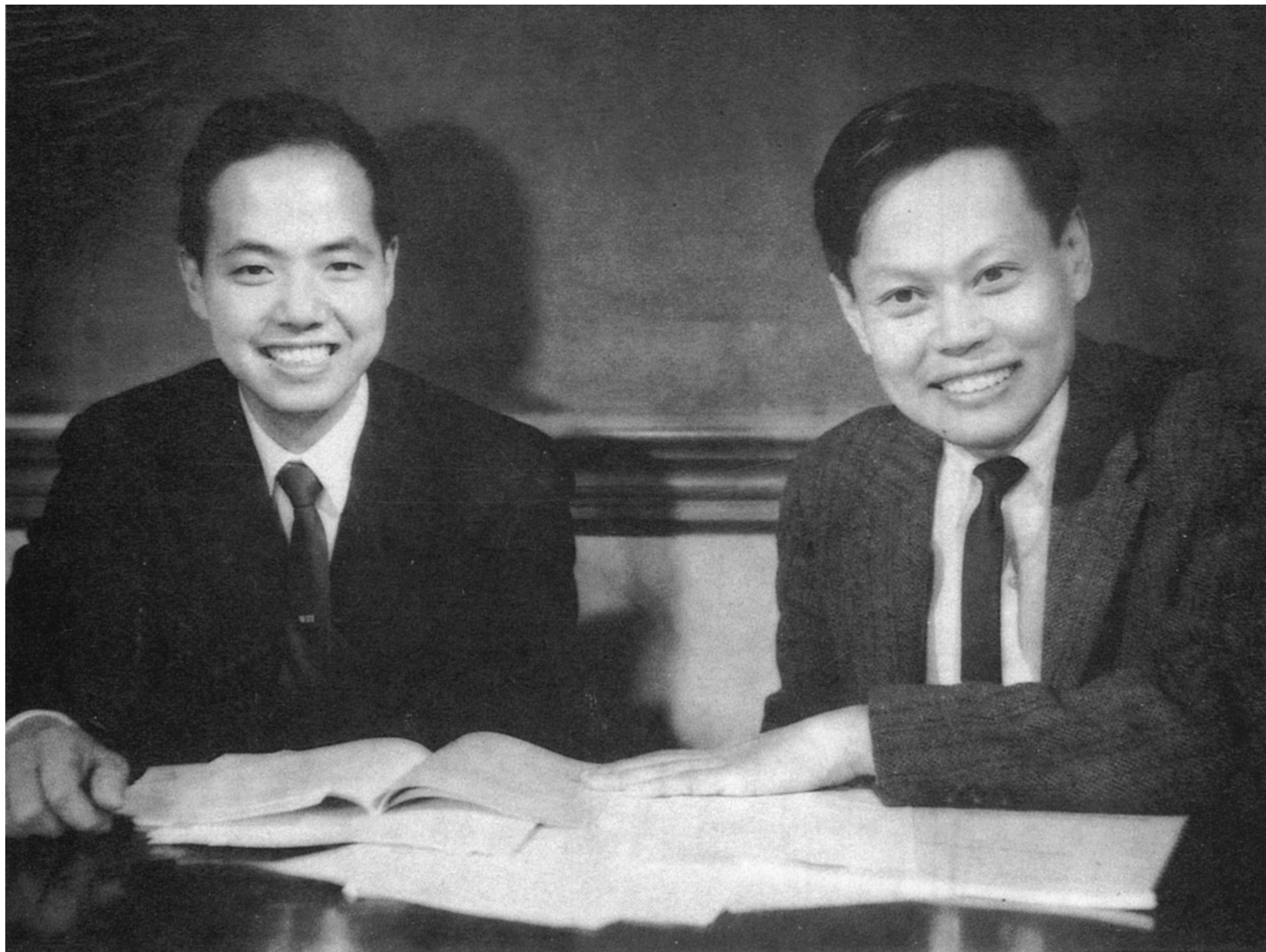
polar & axial vectors



polar



axial
pseudo vector



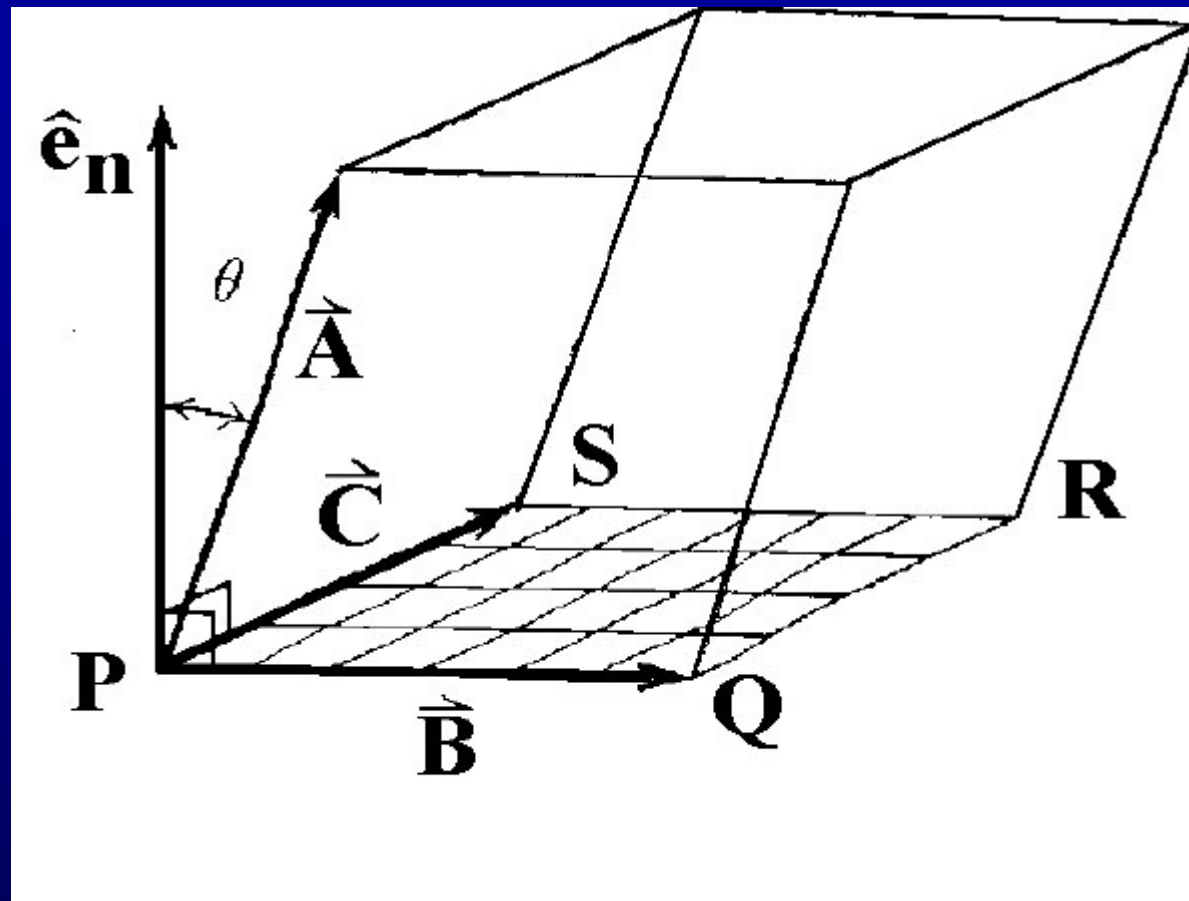
5. Triple products

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

$$A \cdot (B \times C) = C \cdot (A \times B) = B \cdot (C \times A) \\ = -B \cdot (A \times C)$$

mixed product
scalar triple product
box product

cyclic permutation



Volume of the parallelepiped

Example 1.3

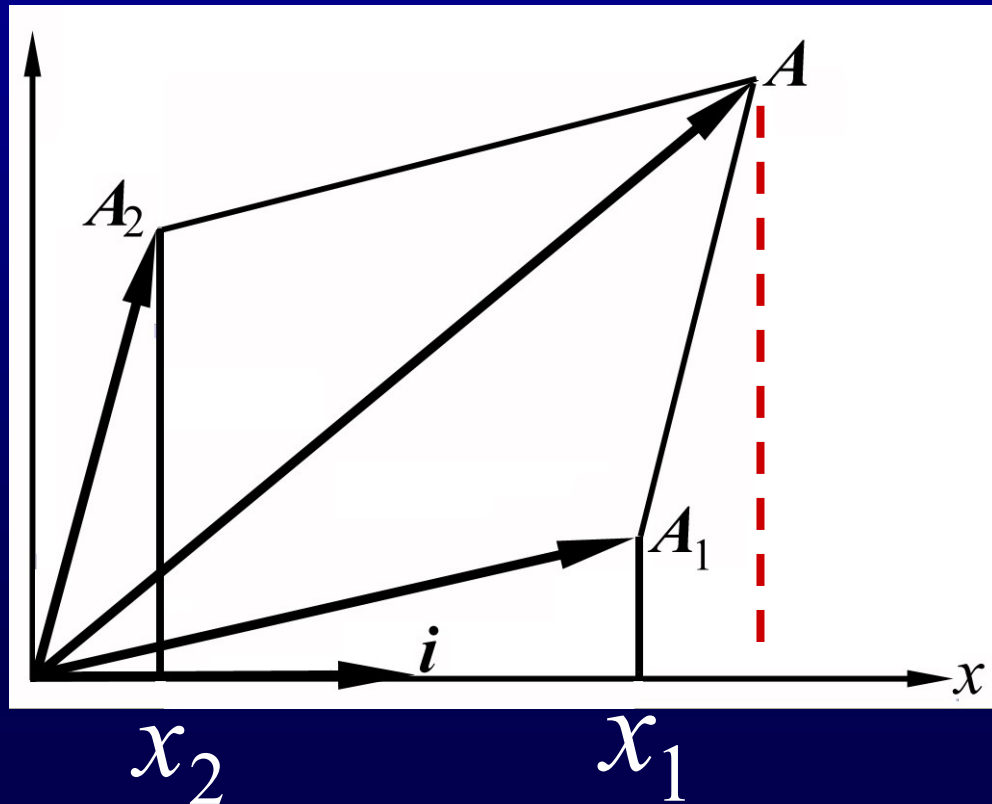
$$x_1 = A_1 \cos(\omega t + \varphi_1)$$

$$x_2 = A_2 \cos(\omega t + \varphi_2)$$

$$x_1 + x_2 = A_1 \cdot i + A_2 \cdot i$$

$$= (A_1 + A_2) \cdot i$$

$$\equiv A \cos(\omega t + \delta)$$



phasor diagrams

AC circuit

(相量图)

illegal: $\frac{1}{A}$, $\ln B$, $\sqrt{C}$, $\exp(D)$, $A = B$

Assignment: 1.1, 1.3/1.4, 1.5, 1.8, 1.9*, 1.12,
1.14, 1.17, 1.19*, 1.22

See § 1.5

* Optional

A/B means A or B

1.5 Orthogonal Coordinate systems

正交

base vector

basis

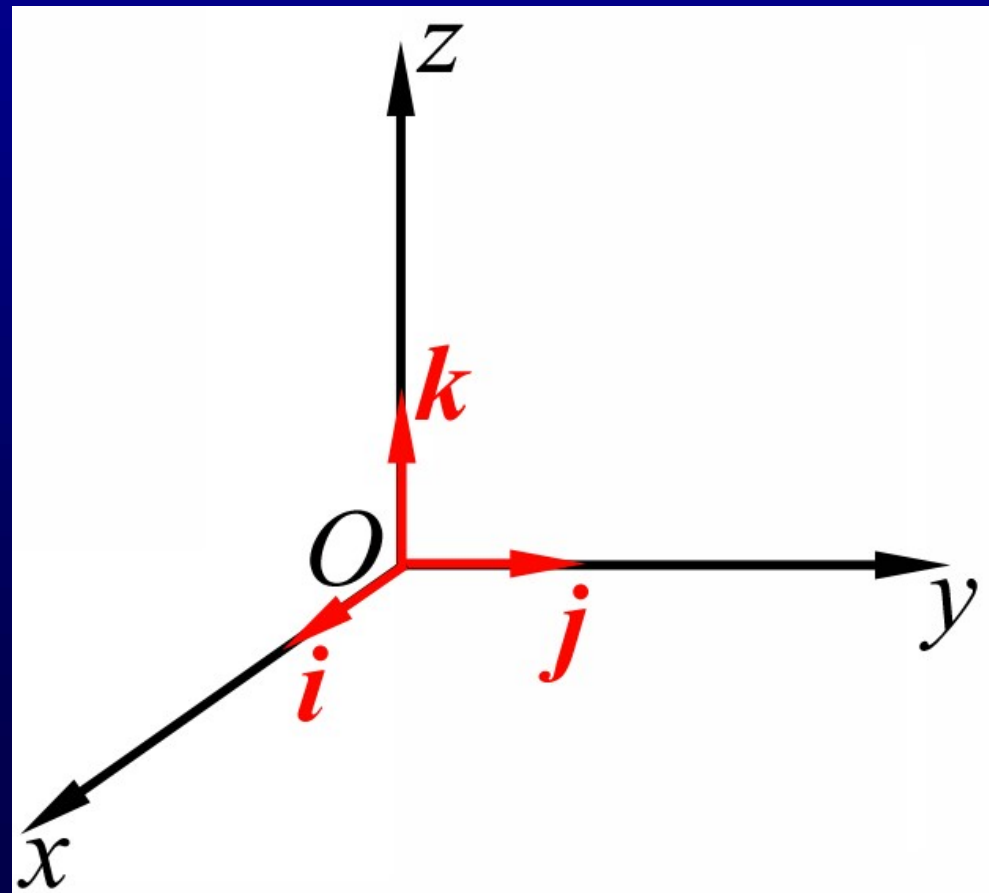
Cartesian system(i, j, k)

$(e_x \ e_y \ e_z)$



Rene Descartes
(1596-1690)

Google: 2,570,000



1.5 Orthogonal Coordinate systems

Cartesian system(i, j, k)

$$i \cdot j = j \cdot k = k \cdot i = 0, \quad \text{orthogonal}$$

$$i \cdot i = j \cdot j = k \cdot k = 1, \quad \text{normalized}$$

归一化

$$\left. \begin{aligned} i \times j &= k, \\ j \times k &= i, \\ k \times i &= j \end{aligned} \right\} \quad \text{right-handed screw}$$

cyclic permutation

Using the properties of base vectors, we have

$$1. \quad A = A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} \quad \cdot \mathbf{i} \quad \cdot \mathbf{j} \quad \cdot \mathbf{k}$$

$$A_1 = A \cdot \mathbf{i}, \quad A_2 = A \cdot \mathbf{j}, \quad A_3 = A \cdot \mathbf{k}$$

$$\begin{aligned} 2. \quad A \cdot B &= (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}) \cdot (B_1 \mathbf{i} + B_2 \mathbf{j} + B_3 \mathbf{k}) \\ &= A_1 B_1 + A_2 B_2 + A_3 B_3 \end{aligned}$$

3.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

where

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = a_{11} \begin{vmatrix} b_{22} & b_{23} \\ c_{32} & c_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} b_{21} & b_{23} \\ c_{31} & c_{33} \end{vmatrix} + a_{13} \begin{vmatrix} b_{21} & b_{23} \\ c_{31} & c_{33} \end{vmatrix}$$

$$\begin{vmatrix} b_{22} & b_{23} \\ c_{32} & c_{33} \end{vmatrix} = b_{22}c_{33} - b_{23}c_{32}$$

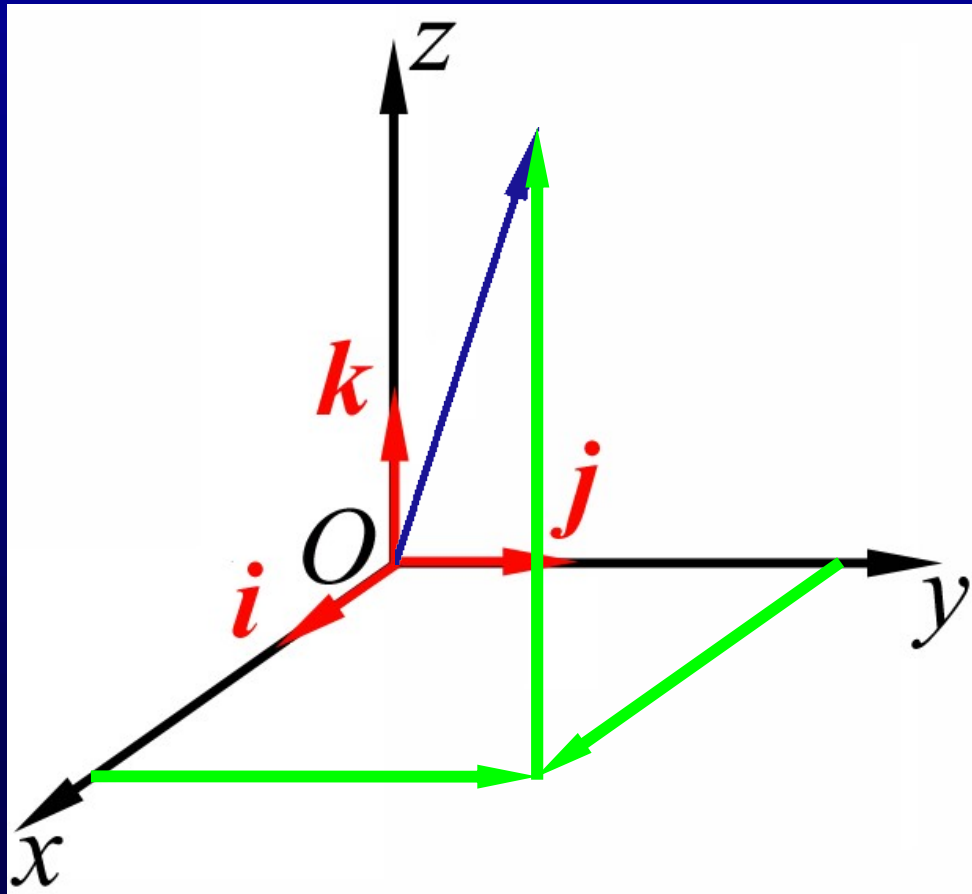
Position vector r

coordinates of end point

(x, y, z)

components

(x, y, z)





Base vectors must be linearly **independent** and **complete**.

orthogonal or oblique?

right-handed or left-handed?

Either OK

A set of vectors are said to be **linearly independent** provided no equation

$$C_1 A_1 + C_2 A_2 + \cdots + C_n A_n = \mathbf{0}$$

holds except the trivial one with

$$C_1 = C_2 = \cdots = C_n = 0.$$

With zero vector in it, a set is linearly dependent.

E.g. for vectors A and B ,

$$C_1A + C_2B = 0, \quad C_1, C_2 \text{ finite}$$

they are linearly dependent. Indeed

$$A = -(C_2/C_1) B$$

So two linearly dependent vectors are collinear.

In a particular space
there exist n linearly independent vectors
but no set of $n + 1$ linearly independent ones,
the space is said to be
 n -dimensional.

Completeness requires the number of base
vectors equal to the dimensionality of space.

Polar coordinate systems

planar ~ $e_\rho \cdot e_\rho = e_\varphi \cdot e_\varphi = 1, \quad e_\rho \cdot e_\varphi = 0$

$$A = A_\rho e_\rho + A_\varphi e_\varphi$$

position vector:

$$\boldsymbol{\rho} = \rho e_\rho$$

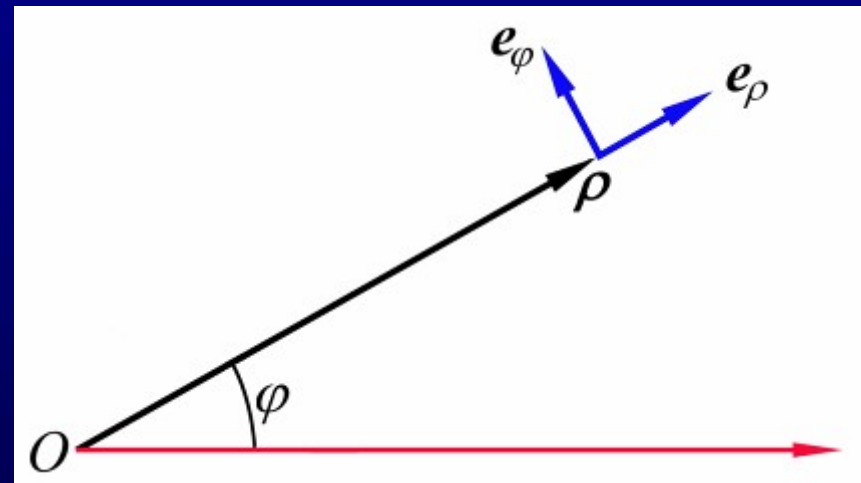
coordinates of end point

$$(\rho, \varphi)$$

components

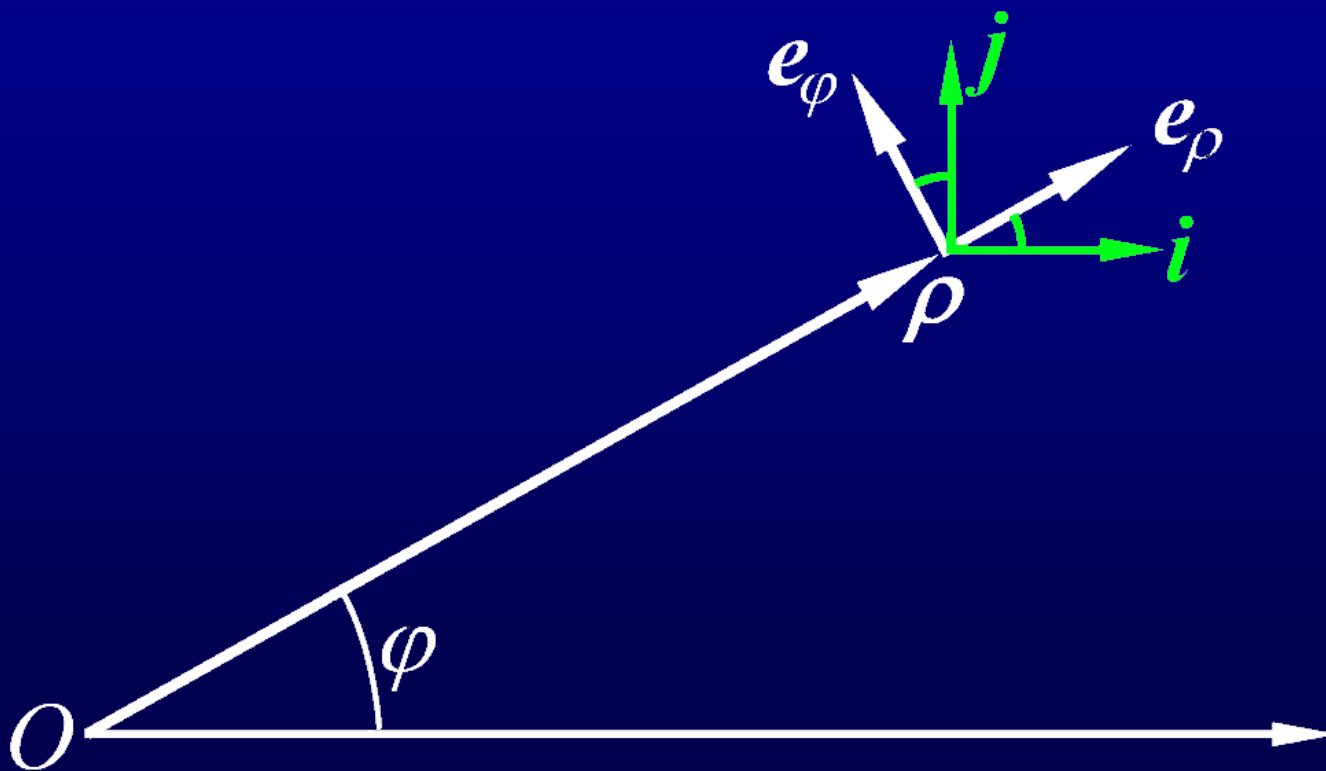
$$(\rho, 0)$$

unit radial vector unit transverse vector



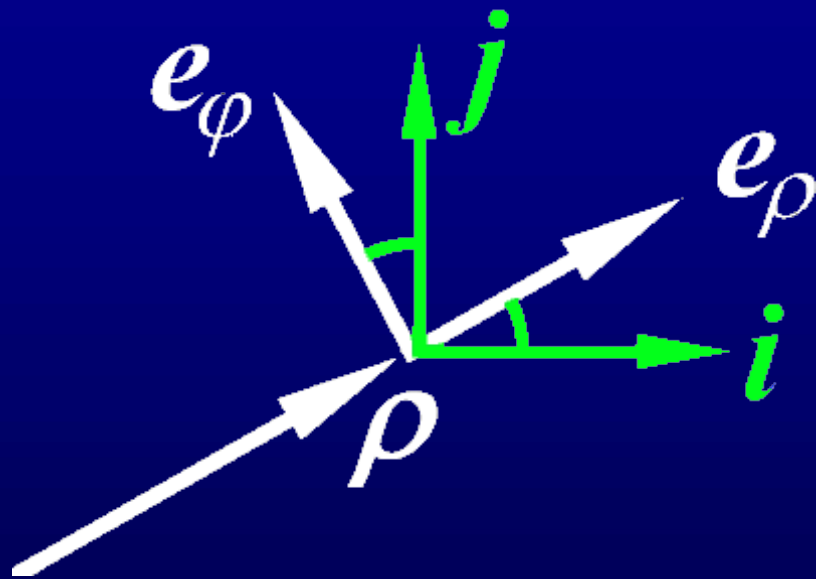
$$e_\rho = \cos \varphi i + \sin \varphi j$$

$$e_\varphi = -\sin \varphi i + \cos \varphi j$$



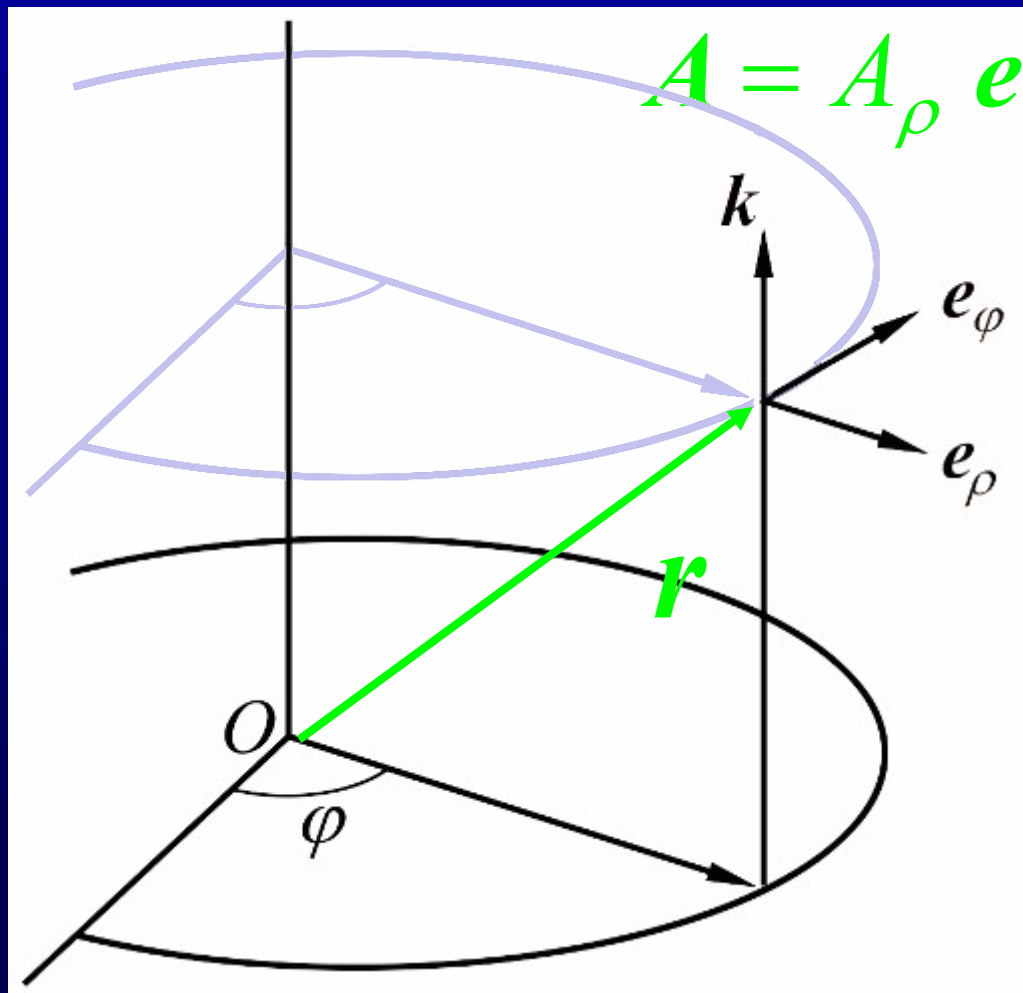
$$e_\rho = \cos \varphi i + \sin \varphi j$$

$$e_\varphi = -\sin \varphi i + \cos \varphi j$$



Cylindrical ~

$(\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{k})$



$$\mathbf{A} = A_\rho \mathbf{e}_\rho + A_\varphi \mathbf{e}_\varphi + A_z \mathbf{k}$$

Position vector

$$\mathbf{r} = \rho + z\mathbf{k}$$

coordinates of end
 (ρ, φ, z)

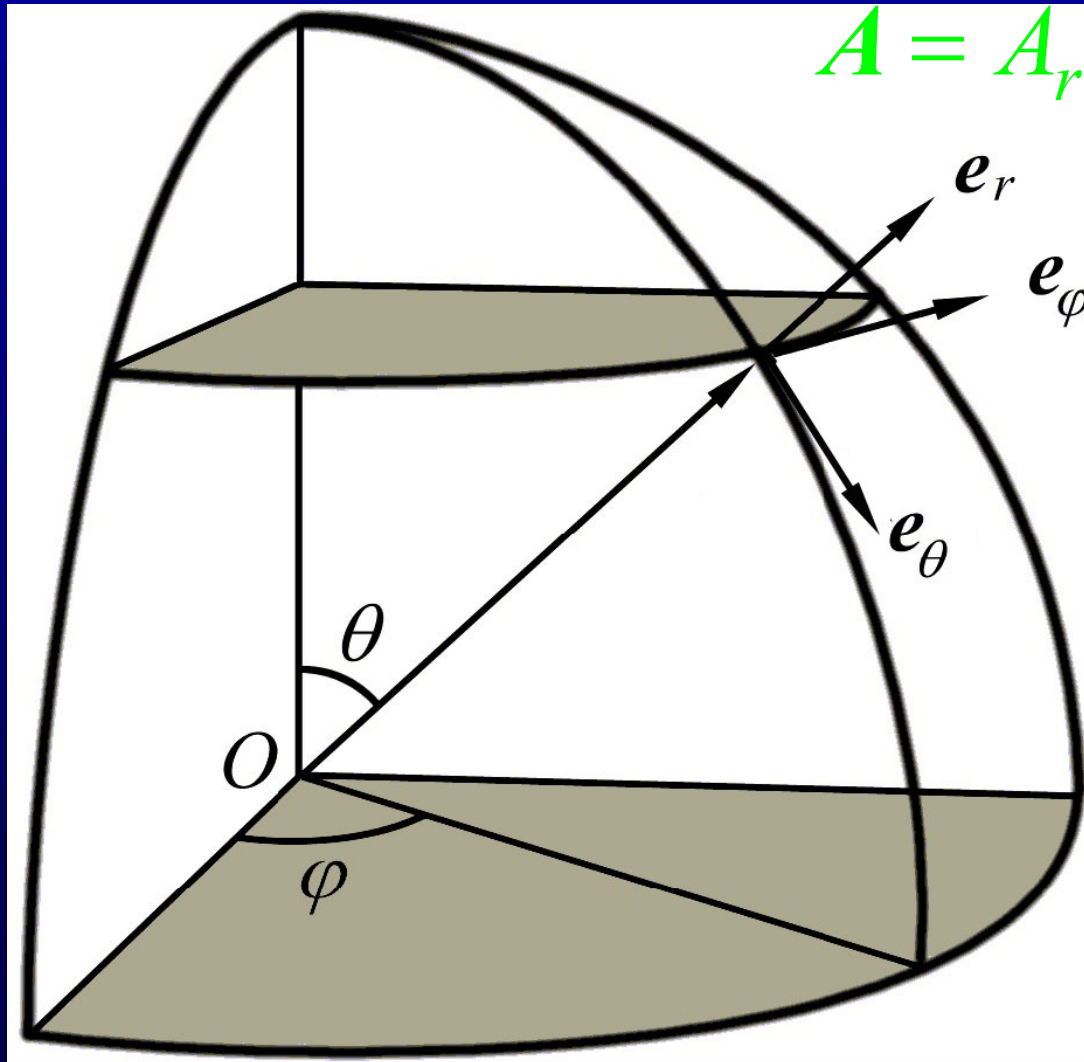
components

$$(\rho, 0, z)$$

Spherical ~

$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$

$$\mathbf{A} = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\varphi \mathbf{e}_\varphi$$



Position vector

$\mathbf{r}$

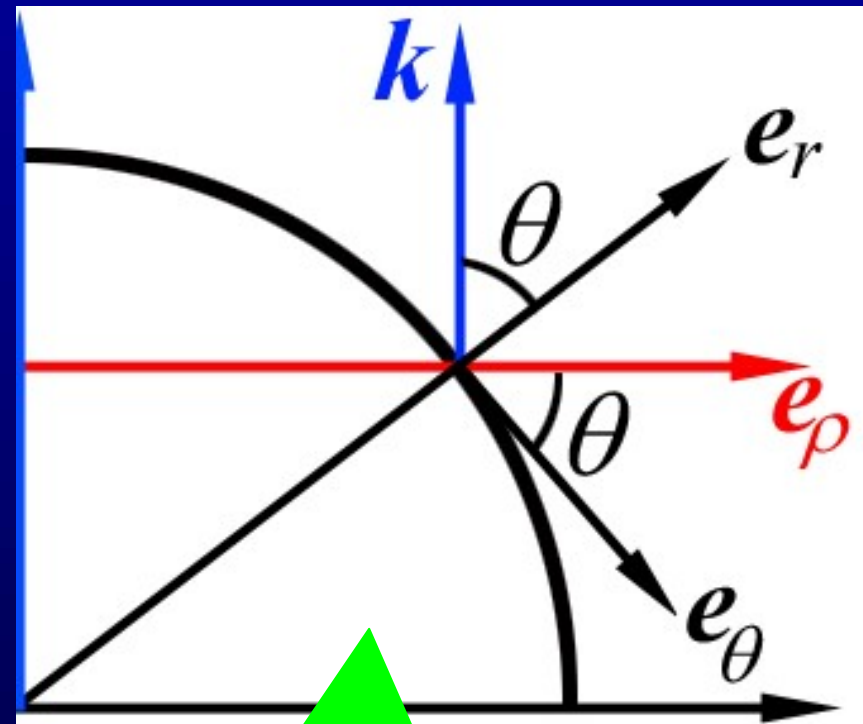
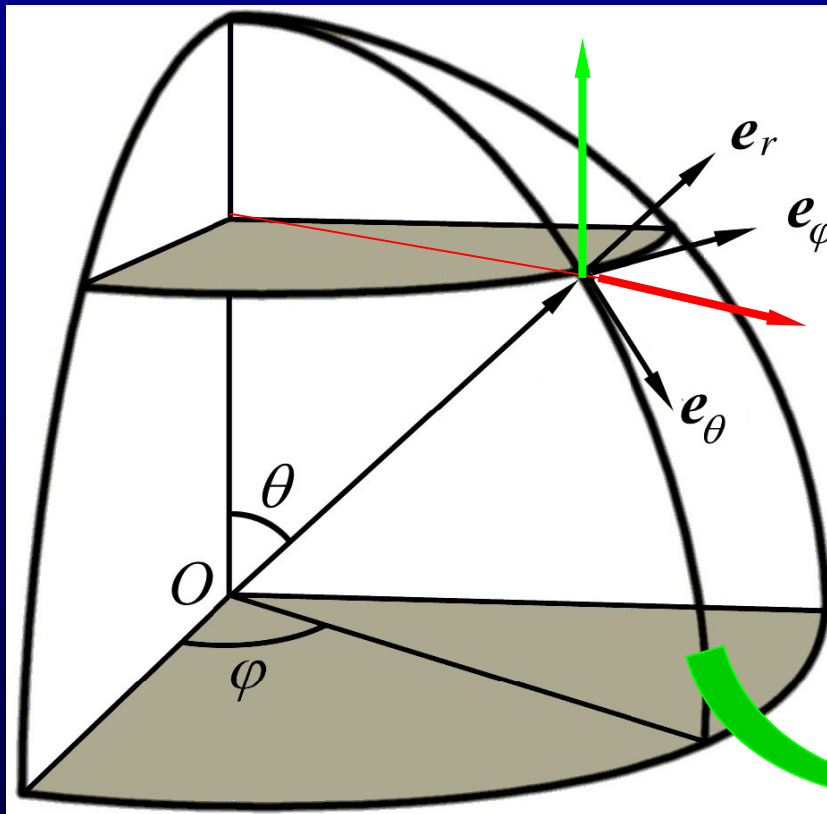
coordinates of end
 (r, θ, φ)

components
 $(r, 0, 0)$

$$\mathbf{e}_r = \cos \theta \mathbf{k} + \sin \theta (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{j})$$

$$\mathbf{e}_\theta = -\sin \theta \mathbf{k} + \cos \theta (\cos \varphi \mathbf{i} + \sin \varphi \mathbf{j})$$

$$\mathbf{e}_\varphi = -\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j}$$



Derivation against *time*

$$\rho = \rho(t), \quad \varphi = \varphi(t)$$

$$\mathbf{e}_\rho = \cos \varphi \mathbf{i} + \sin \varphi \mathbf{j}$$

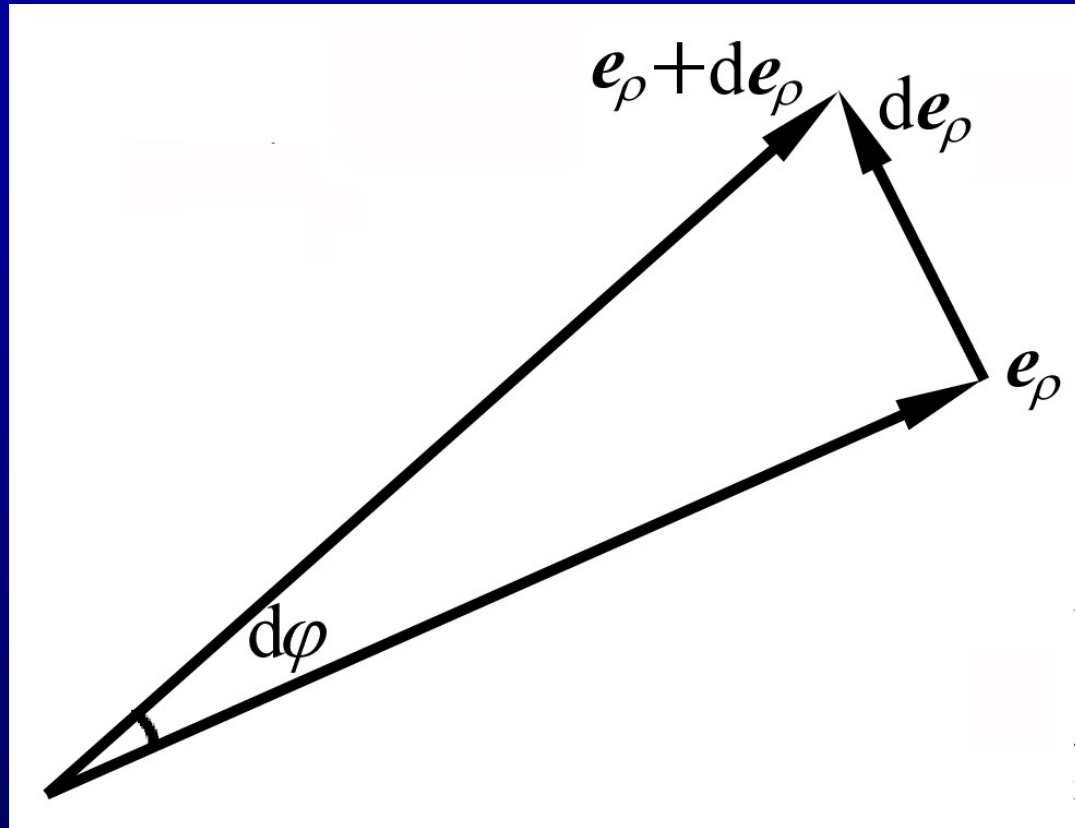
$$\mathbf{e}_\varphi = -\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j}$$

$$\frac{d\mathbf{e}_\rho}{dt} = (-\sin \varphi \mathbf{i} + \cos \varphi \mathbf{j})\dot{\varphi} = \dot{\varphi} \mathbf{e}_\varphi$$

$$\frac{d\mathbf{e}_\varphi}{dt} = (-\cos \varphi \mathbf{i} - \sin \varphi \mathbf{j})\dot{\varphi} = -\dot{\varphi} \mathbf{e}_\rho$$

$$d\mathbf{e}_\rho = 1 \cdot d\varphi \mathbf{e}_\varphi$$

$$\frac{d\mathbf{e}_\rho}{dt} = \dot{\varphi} \mathbf{e}_\varphi$$



$$\boldsymbol{\rho} = \rho \mathbf{e}_\rho$$

$$\dot{\boldsymbol{\rho}} = \dot{\rho} \mathbf{e}_\rho + \rho \dot{\phi} \mathbf{e}_\phi$$

$$\ddot{\boldsymbol{\rho}} = (\ddot{\rho} - \rho \dot{\phi}^2) \mathbf{e}_\rho + (\rho \ddot{\phi} + 2\dot{\rho} \dot{\phi}) \mathbf{e}_\phi$$

$$\frac{d\mathbf{e}_\rho}{dt} = \dot{\phi} \mathbf{e}_\phi$$

$$\frac{d\mathbf{e}_\phi}{dt} = -\dot{\phi} \mathbf{e}_\rho$$

In circular motion

$$\dot{\rho} = \ddot{\rho} = 0$$

$$\dot{\boldsymbol{\rho}} = \rho \dot{\phi} \mathbf{e}_\phi$$

$$\ddot{\boldsymbol{\rho}} = -\dot{\phi}^2 \rho + \rho \ddot{\phi} \mathbf{e}_\phi$$

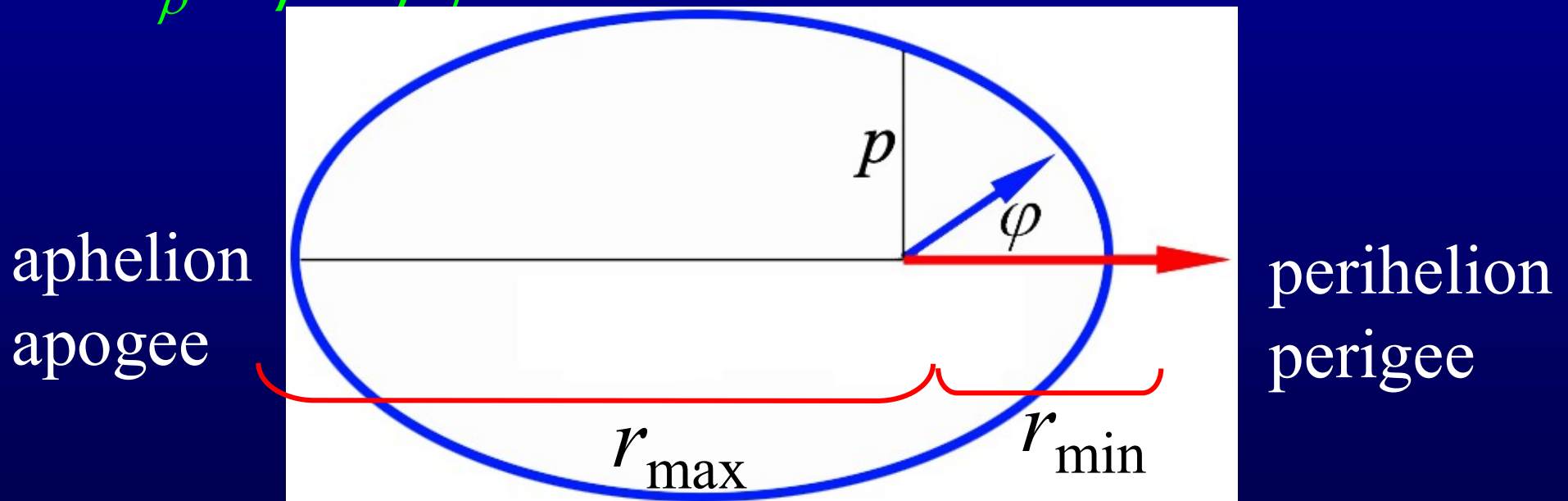
*In **uniform** **circular** **motion***
 $\dot{\phi} = \text{constant}$

Example 1.5

$$\rho = \frac{p}{1 + e \cos \varphi} \quad (0 < e < 1)$$

Find the radial acceleration at $\varphi = 0, \pi$

$$a_\rho = \ddot{\rho} - \rho \dot{\varphi}^2$$



radius pl. radii

$$a_{\rho} = -\frac{p}{(1+e)^2} \dot{\phi}^2 = -\frac{r_{\min}^2 \dot{\phi}^2}{p}, \quad \varphi = 0$$

$$a_{\rho} = -\frac{p}{(1-e)^2} \dot{\phi}^2 = -\frac{r_{\max}^2 \dot{\phi}^2}{p}, \quad \varphi = \pi$$

$$\ddot{\mathbf{r}} = (\ddot{\rho} - \rho\dot{\varphi}^2)\mathbf{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\mathbf{e}_\varphi \Rightarrow a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2$$

$$\rho = \frac{p}{1 + e \cos \varphi} \quad \dot{\rho} = \frac{pe \sin \varphi \dot{\varphi}}{(1 + e \cos \varphi)^2}$$

$$\ddot{\rho} = \frac{2pe^2 \sin^2 \varphi \dot{\varphi}^2}{(1 + e \cos \varphi)^3} + \frac{pe \cos \varphi \dot{\varphi}^2}{(1 + e \cos \varphi)^2}. \quad (\ddot{\varphi} = 0)$$

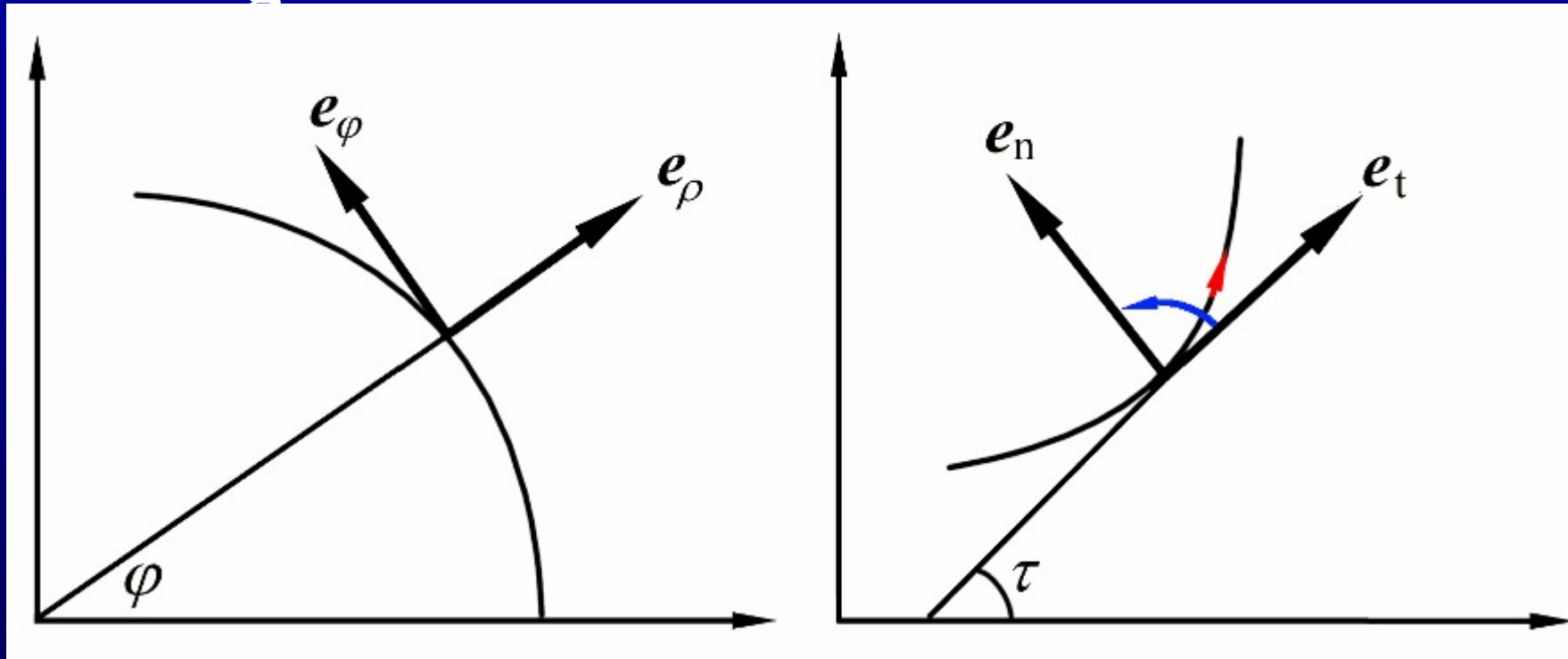
$$a_\rho = \left[\frac{2e^2 \sin^2 \varphi}{(1 + e \cos \varphi)^3} + \frac{e \cos \varphi}{(1 + e \cos \varphi)^2} - \frac{1}{1 + e \cos \varphi} \right] p \dot{\varphi}^2$$

$$a_\rho = -\frac{p}{(1 + e)^2} \dot{\varphi}^2 = -\frac{r_{\min}^2 \dot{\varphi}^2}{p}, \quad \varphi = 0$$

$$a_\rho = -\frac{p}{(1 - e)^2} \dot{\varphi}^2 = -\frac{r_{\max}^2 \dot{\varphi}^2}{p}, \quad \varphi = \pi$$

内乗

*intrinsic coordinate system
tangential and normal directions



(s, τ)

$$\tau = f(s)$$

*intrinsic equation
of the curve*

$$\mathbf{v} = \frac{ds}{dt} \mathbf{e}_t = v \mathbf{e}_t$$

$$\mathbf{a} = \frac{d}{dt}(v \mathbf{e}_t) = \frac{dv}{dt} \mathbf{e}_t + v \frac{d\mathbf{e}_t}{d\tau} \frac{d\tau}{ds} \frac{ds}{dt}$$

$$= \frac{dv}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n$$

radius of curvature

curvature

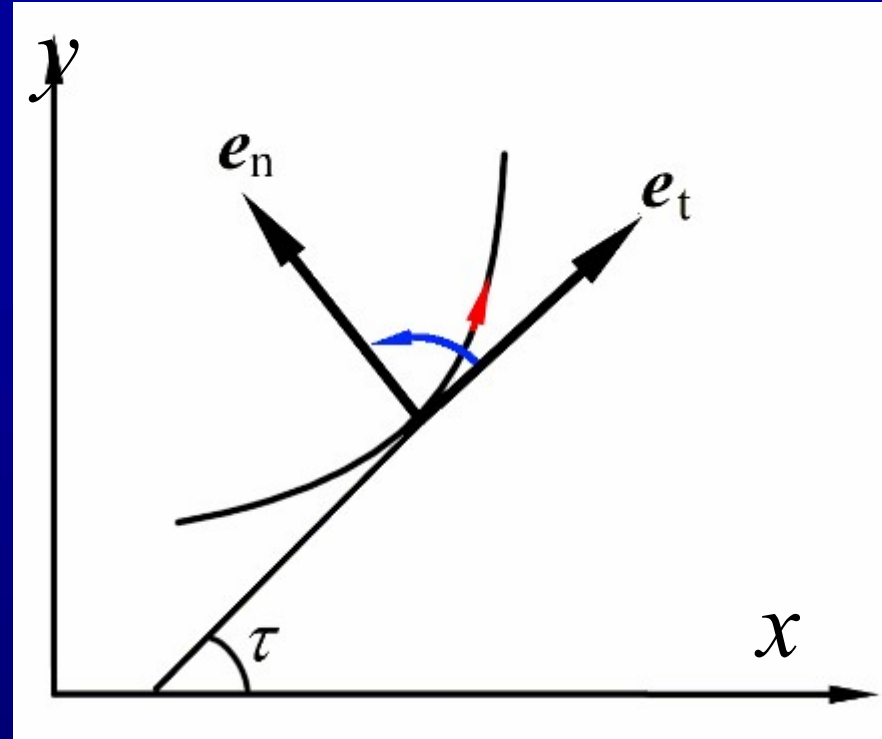
$$\kappa = \frac{d\tau}{ds} = \frac{\frac{d^2 y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{1}{\rho}$$

osculating circle
密切圆
inscribed circle

$$\mathbf{e}_t = \cos \tau \mathbf{i} + \sin \tau \mathbf{j}$$

$$\mathbf{e}_n = -\sin \tau \mathbf{i} + \cos \tau \mathbf{j}$$

$$\frac{d\mathbf{e}_t}{d\tau} = -\sin \tau \mathbf{i} + \cos \tau \mathbf{j} = \mathbf{e}_n.$$



$$\frac{d\mathbf{e}_t}{ds} = -\sin \tau \frac{d\tau}{ds} \mathbf{i} + \cos \tau \frac{d\tau}{ds} \mathbf{j}$$

$$\left| \frac{d\mathbf{e}_t}{ds} \right| = \sqrt{\left(-\sin \tau \frac{d\tau}{ds} \right)^2 + \left(\cos \tau \frac{d\tau}{ds} \right)^2} = \frac{d\tau}{ds} = \kappa$$

$$\tan \tau = y'$$

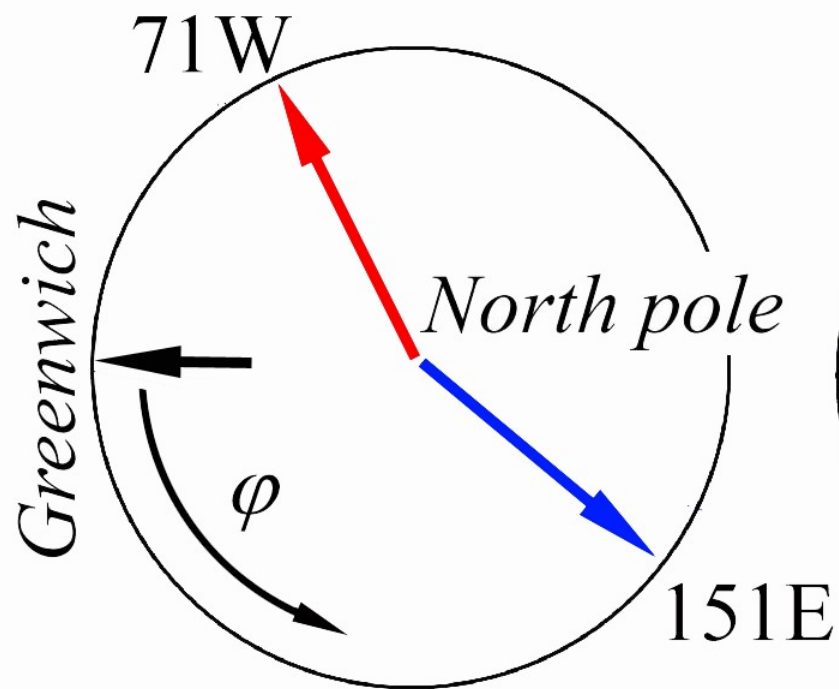
$$\tau = \arctan y'$$

$$d\tau = (\arctan y')' dx = \frac{y''}{1 + (y')^2} dx$$

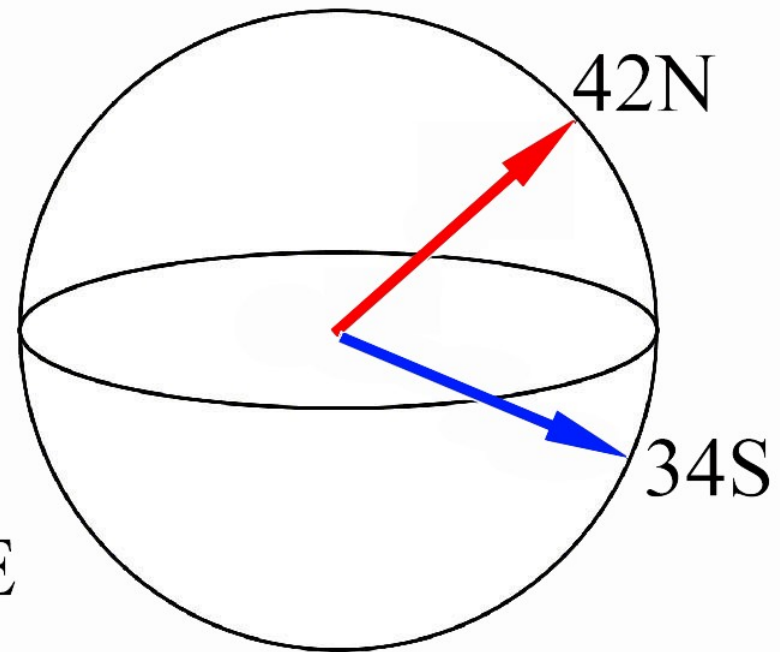
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + y'^2} dx$$

$$\kappa = \frac{d\tau}{ds} = \frac{\frac{d^2 y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{1}{\rho}$$

Assignment:1.21



longitude



latitude

Frenet–Serret Frame

$\mathbf{t}$ unit tangent vector 单位切矢量

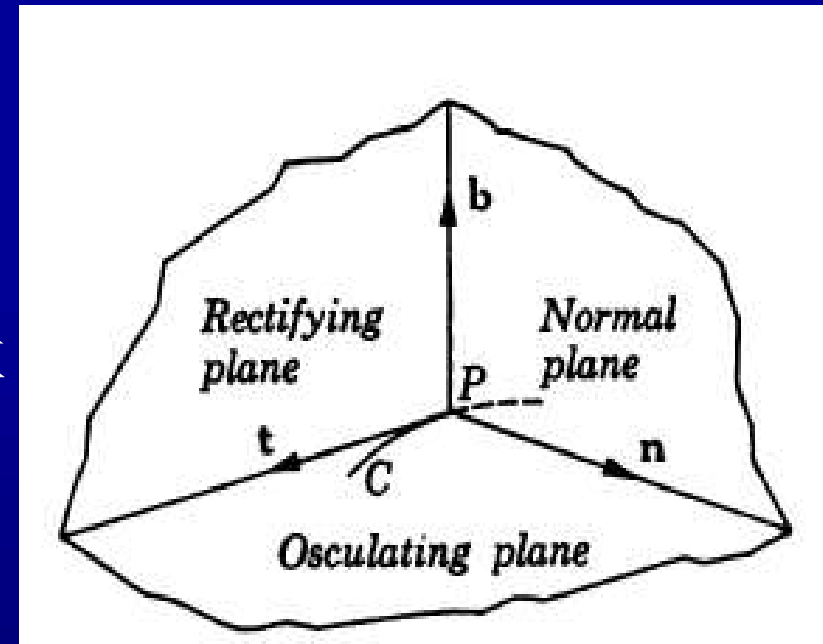
$\mathbf{n}$ unit normal vector 单位法矢量

$\mathbf{b}$ unit binormal vector 单位从法矢量

osculating plane 密切平面

normal plane 法平面

rectifying plane 从法平面



$\mathbf{r} = \mathbf{r}(s)$, s – arc length. $\mathbf{t}(s) = \mathbf{r}'(s)$, $\kappa = |\mathbf{t}'(s)| = |\mathbf{r}''(s)|$ – curvature.

$$\mathbf{t}(s) \cdot \mathbf{t}(s) = \mathbf{r}'(s) \cdot \mathbf{r}'(s) = 1 \Rightarrow \mathbf{t}(s) \cdot \mathbf{t}'(s) = \mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0.$$

$$\mathbf{n}(s) = \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|} = \frac{\mathbf{r}''(s)}{\kappa}. \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s). \quad \tau(s) = |\mathbf{b}'(s)| - \text{torsion 挠率}$$

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \frac{d\mathbf{b}}{ds} = -\tau \mathbf{n}.$$

Frenet formulas

$\mathbf{b} = \mathbf{t} \times \mathbf{n} \Rightarrow \mathbf{n} = \mathbf{b} \times \mathbf{t}, \mathbf{t} = \mathbf{n} \times \mathbf{b}$ cyclic permutation

$\mathbf{b} \cdot \mathbf{t} = 0, \mathbf{b} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{b}' \cdot \mathbf{t} + \mathbf{b} \cdot \mathbf{t}' = 0 \quad (\mathbf{t}' = \kappa \mathbf{n}), \mathbf{b}' \cdot \mathbf{t} = 0.$

$\mathbf{b} \cdot \mathbf{b} = 1 \Rightarrow \mathbf{b}' \cdot \mathbf{b} = 0. \quad \mathbf{b}' \cdot \mathbf{t} = 0. \mathbf{b}' \cdot \mathbf{b} = 0 \Rightarrow \mathbf{b}' \parallel \mathbf{n}.$

$\mathbf{b}' = -\tau \mathbf{n}$ (definition) $\Rightarrow |\tau(s)| = |\mathbf{b}'(s)|.$

$\mathbf{n}(s) = \mathbf{b}(s) \times \mathbf{t}(s) \Rightarrow \mathbf{n}'(s) = \mathbf{b}'(s) \times \mathbf{t}(s) + \mathbf{b}(s) \times \mathbf{t}'(s)$
 $= -\tau \mathbf{n}(s) \times \mathbf{t}(s) + \mathbf{b}(s) \times \kappa \mathbf{n}(s) = \tau \mathbf{b}(s) - \kappa \mathbf{t}(s).$

$\mathbf{b} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{b}' \cdot \mathbf{n} + \mathbf{b} \cdot \mathbf{n}' = 0,$

$\tau(s) = -\mathbf{b}' \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}' = \mathbf{n}' \cdot (\mathbf{t} \times \mathbf{n}).$

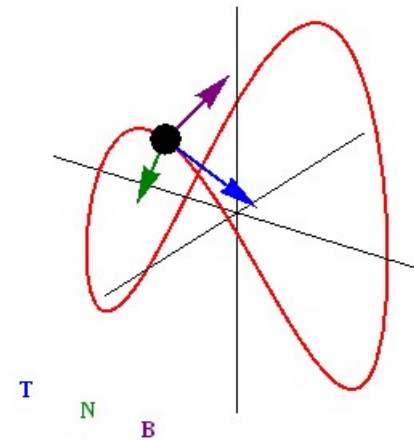
$$= \mathbf{t} \cdot (\mathbf{n} \times \mathbf{n}') = \frac{\mathbf{r}' \cdot (\mathbf{r}'' \times \mathbf{r}''')}{\kappa^2}.$$

For $\mathbf{r} = \mathbf{r}(t)$:
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\tau(t) = \frac{\mathbf{r}'(t) \cdot (\mathbf{r}''(t) \times \mathbf{r}'''(t))}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}.$$

Proof of Frenet Formulas

Moving Frenet bases



A proof of the Lagrange's formula for vector triple product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Proof: If the vectors $\mathbf{B}$ and $\mathbf{C}$ are collinear, then it is obvious that both sides of above equation are zero. Therefore, the above relationship holds in this case.

If $\mathbf{B}$ and $\mathbf{C}$ are not collinear, then the product $\mathbf{B} \times \mathbf{C}$ is orthogonal to both $\mathbf{B}$ and $\mathbf{C}$ and it is a vector that orthogonal to the plane containing $\mathbf{B}$ and $\mathbf{C}$. Therefore, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is orthogonal to $\mathbf{B} \times \mathbf{C}$ and then in the plane containing $\mathbf{B}$ and $\mathbf{C}$. Thus, we have

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha \mathbf{B} - \beta \mathbf{C}$$

where α and β are to be determined. Since

$$\mathbf{A} \cdot [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] = \alpha(\mathbf{A} \cdot \mathbf{B}) - \beta(\mathbf{A} \cdot \mathbf{C}) = 0,$$

we have $\alpha(\mathbf{A} \cdot \mathbf{B}) = \beta(\mathbf{A} \cdot \mathbf{C})$ or $\frac{\alpha}{\mathbf{A} \cdot \mathbf{C}} = \frac{\beta}{\mathbf{A} \cdot \mathbf{B}}$.

Letting $\frac{\alpha}{A \cdot C} = \frac{\beta}{A \cdot B} \equiv \lambda$, we obtain

$$A \times (B \times C) = \lambda [B(A \cdot C) - C(A \cdot B)].$$

Taking $A = i$, $B = k$ and $C = i$

Left side:

$$A \times (B \times C) = i \times (k \times i) = i \times j = k;$$

Right side

$$\lambda [B(A \cdot C) - C(A \cdot B)] = \lambda [k(i \cdot i) - i(i \cdot k)] = \lambda k.$$

Therefore, $\lambda = 1$.